Performance Comparison of LDPC Codes Generated With Various Code-Construction Methods

Zsolt Polgar, Florin Ardelean, Mihaly Varga, Vasile Bota

***Abstract*—Finding good LDPC codes for high speed mobile transmissions is a difficult task due to the rate and code length restrictions imposed by these transmissions. The paper proposes a BER vs. SNR performance comparison of LDPC codes generated with several code construction algorithms, the purpose of this comparison being the selection of code constructions algorithms which ensure good performances for some imposed length and code-rate restrictions. The paper also proposes a geometrical LDPC code construction with girth 8, small number of length-8 loops and variable length of the codeword for a given rate, construction that can be used especial for coding rates ≤ 0.6 .**

***Index Terms*—LDPC codes, code construction algorithms, girth of the Tanner graph, BER/SNR performances**

I. Introduction

ENSURING high throughput and/or low bit error probability mobile transmissions over a large range of signal/noise ratios requires adaptive change of the QAM-constellation and forward correction coding rate. Moreover, for the mobile transmissions, the channel state prediction issues and the decrease of the latency inserted, impose the codeword length to be rather small, usually smaller than 1000 bits [13], depending on the system configuration. Due to the high coding gains ensured with rather high coding rates and to the acceptable small complexity decoding, the LDPC codes are significant candidates for such transmissions [1] [2].

The performances of the LDPC codes, decoded with the Sum-Product algorithm [2], depend mostly of the following parameters:

- Codeword length – long codeword codes (thousands of bits) ensure better performances than short codeword codes (hundreds of bits), [2].
- The number of bits of „1” in the columns of the control matrix H – the order of the bit-nodes d_b , the number of bits of „1” in the rows of H – the order of the check-nodes d_c [4]; also the way this bits of „1” are distributed affects the correction capability [9]. Constant values of d_b and d_c for all bit and check nodes, lead to better performances. The optimum number of “1” bits in a column of H is shown to be 3 [9].
- The minimum-length loop of the bit-nodes (girth) within the Tanner graph associated to the code, affects the performances, i.e. higher girth ensures better performances, [6] [7]. The girth and the coding rate impose the minimum codeword length.
- The minimum Hamming distance of the code – it has smaller influence when the codes are decoded with the Sum-Product algorithm.

The performances of the array-based LDPC codes [10] in mobile transmissions are presented in [12]. But the performances of these codes are limited by their girth g = 6. In order to increase the correction capability of the LDPC codes, some other ways to build the H matrix that ensure, for a given coding rate and length, a higher girth, should be considered.

The paper focuses on the following questions:

- Comparison between the BER(SNR) performances of regular LDPC codes with the same rate and length and different girth, generated by several construction methods; the girth-values considered are g = 6, 8, 12.
- Presentation and analysis of a LDPC code construction algorithm with g = 8, and with a small number of length-8 loops (girth almost 10) developed by the authors; a performance comparison between these codes and the codes generated by some other code construction methods.

II. Analyzed Code-Construction Methods

The methods to generate the LDPC codes considered in this paper are briefly described below; they are described in detail in the mentioned references.

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A. L(m,q) Codes

A_1	A_2	A_3	……	A_k
B_1	B_2	B_3	……	B_k
C_1	C_2	C_3	……	C_k

Fig. 1 Structure of the L(2,q) and L(3,q) codes H matrices for d_b=3

L(m,q) codes [11] have a codeword length $N = q^m$, where q is a prime number or a power of a prime number and m is a natural number. Due to limitation of the codeword length, only codes with m = 2 or 3 were considered. The control matrix H of these codes is generated by removing a number of rows, according to the coding rate desired, out of a square matrix M ($q^m \times q^m$). Each row of matrix M is composed of q square sub-matrices of ($q^{m-1} \times q^{m-1}$) elements each; these sub-matrices are obtained by permutations from a basis sub-matrix. The L(2,q) codes are identical to the a class of array-codes [10]. To generate a regular code, that usually provides better performances, with a coding rate $R_c = 1 - k/q$, only $k \cdot q^{m-1}$ rows should be retained from the complete matrix; k should equal 3 if a bit-node order $d_b = 3$ is to be ensured. The first two rows of table 1 present the minimum code word lengths of the L(2,q) and L(3,q) codes for several coding rates and $d_b = 3$. The structure of the H matrix of these codes is presented in fig. 1, the B_i and C_i matrices being permutations of the A_i matrices. In the case of L(2,q), the A_i are unitary matrices The coding rate may be decreased by shortening the parent block code; the regular character of the code should be maintained, if possible.

B. Codes Generated by Combinatorial Constructions

The most studied constructions of this type, [3] [4] [8], are based on the Kirkman triple systems [15], a particular case of the Steiner triple systems [15]. These systems are sets of 3-element combinations taken from a given set, combinations which have some restrictions imposed. These systems allow the generation of codes with $d_b = 3$ and constant order of d_c. Basically, this construction method generates codes of girth = 6, with rather short code words, for high coding rates, see row 3 in table 1. The H control matrix of these codes is the connection matrix between the elements of the considered set and the Kirkman system, i.e. the incidence matrix of the system, [4].

C. Codes Generated by Geometrical Constructions

The geometrical constructions employ the structure-graph associated to the code [5]; this graph, extracted from the Tanner graph, describes only the connections between the check-nodes. By splitting the check nodes into several groups and imposing restrictions for the connections between elements of different groups, codes with a certain girth can be generated. Using this construction method reference [5] presents the construction of LDPC codes with $d_b = 2$ and girth equaling 12, 20 or 24. This paper analyzes only the codes with girth=12, G_12_2 (row 4 in table 1), due to the code length limitations. In [6] and [7] codes with $d_b = 3$ are generated, at the expense of decreasing the girth to 8, G_8_3 (row 5 in table 1). Codes based on the geometrical construction are composed by elementary square matrices with lower bounded dimension. The minimum lengths of G_8_3 and G_12_2 for some coding rates are presented in table 1. Only codes with lengths smaller than 3000 are considered.

Code/R_c	0.33	0.5	0.6	0.66	0.7	0.75	0.777	0.8	0.81	0.84	0.87	0.89	0.93	0.94
L(2,q) - L_c	20*	16	49*		81*	121*	169*		256*	361*	529*	729	1848*	2208*
L(3,q) - L_c	27*	294	343*	512*	1331*	1331	2197*							
Kirkman-L_c					70					249		532	1426	2035*
G_12_2 - L_c	39	60	145	282	469*	760	1125	1670*	2453*					
G_8_3 - L_c		126	216*	495	570	756	949*	1185	1296*	2970*				

Table 1 Minimum code lengths of LDPC codes generated with the studied code construction methods for some imposed rates

* these code lengths correspond only approximately to the presented coding rates

III. BER vs. SNR Performance Comparison between the LDPC Codes Generated With The Studied Construction Methods

In order to establish the construction which ensures the best performances, for a given girth, codes constructed with different methods, but having the same girth and about the same coding rate and length, were evaluated computer simulations using the program and transmission environment described in [14]. The 2PSK modulation, AWGN channel and Sum-Product decoding algorithm with 100 iterations/codeword were employed. The girth was computed using an algorithm implemented by the authors. There should be noted that the same value of the girth does not involve the same girth-distribution and the same distribution of “1” in the columns and in the lines of the H matrix, parameters that also affect the performances of the LDPC codes decoded with the Sum-Product algorithm.

A. Performance Comparison of LDPC Codes With Girth = 6

L(2,q) and Kirkman codes are considered. Figure 2 a. presents the BER vs. SNR of codes having R_c = 0.82 – 0.89 and figure 2.b shows the curves for codes with R_c = 0.65 – 0.70. These results and additional simulations performed show that L(2,q) codes ensure better performances than the Kirkman codes; the difference is about 1 – 1.5 dB, for low bit error rates, for the same coding rate and code length. By comparing the two figures, there should be noticed that the code length has a greater influence upon the coding gain than the code rate, see the R_c = 0.65 and L_c = 72 codes of fig.2.b, compared to the R_c = 0.85 and significantly longer codes of figure 2.a.

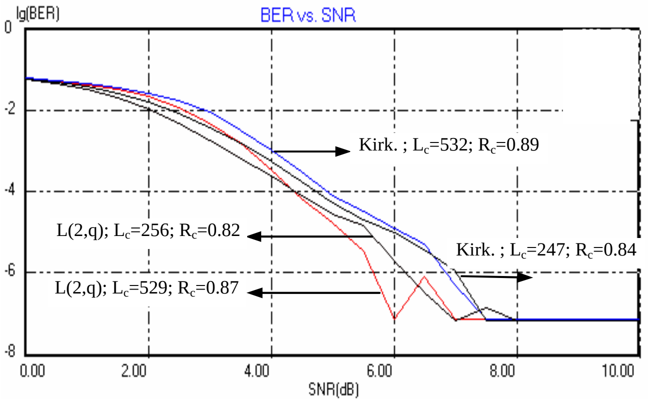


Fig. 2.a. BER vs. SNR of codes with girth=6; high R_c

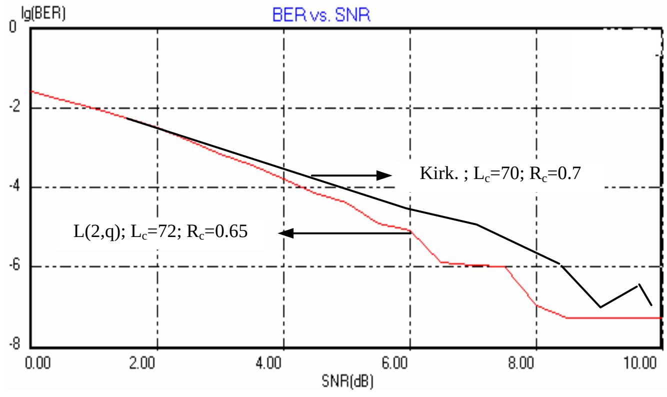


Fig. 2.b. BER vs. SNR of codes with girth=6; medium R_c

B. Performance Comparison of LDPC Codes With Girth Equaling 8 or 12

In this case, the L(3,q), G_8_3 (g = 8) and G_12_2 (g = 12) codes were considered. The BER vs. SNR performances of these codes, having about the same rate and length, are presented in figure 3.a. for codes with L_c = 294 and R_c = 0.5 and in fig. 3.b. for codes with L_c = 2028 and R_c = 0.75.

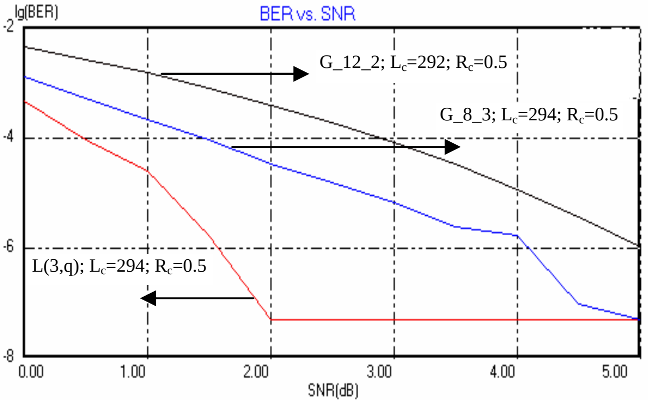


Fig. 3.a. BER vs. SNR of codes with girth = 8 or 12, L_c = 294; R_c = 0.5

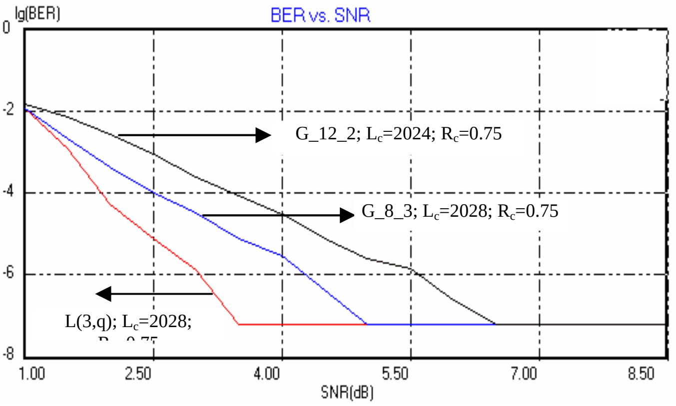


Fig. 3.b. BER vs. SNR of codes with girth = 8 or 12, L_c = 2028; R_c = 0.75

These results and additional simulations lead to the following conclusions:

- for long code words, the L(3,q) codes have better performances than the G_8_3 codes; the improvement is about 1.5 – 2 dB at low BER, for an imposed rate and code word length;
- for short code words, the L(3,q) codes have better performances than the G_8_3 codes with about 2.5 dB at low BER, for an imposed rate and code word length;
- the G_8_3 code have better performances than the G_12_2 codes, with about 1.5 dB at low BERs, for the same R_c and L_c. The poorer performances of the G_12_2 codes, which have bigger girth, could be explained by the smaller order of the bit nodes, i.e. for the G_12_2 d_b = 2 and for G_8_3 d_b = 3.
- the length of the code word has about the same influence upon the coding gain, as the coding rate.

Even if G_8_3 codes have poorer performances than L(3,q) codes, the minimum length of this codes for a given rate is smaller than the length of L(3,q) codes, and this length can be changed more easily. The structure of the G_8_3 codes control matrix H is that presented in fig.1, the A_i matrices being unitary. The dimension of the elementary matrices is lower bounded and there are no other restrictions concerning the dimension of these matrices, while the dimension of the L(3,q) elementary matrices is (q*q), q prim number or power of a prime number

IV. Codes With Girth Almost 10

Better performances than those of L(3,q) codes (with girth 8) can be obtained by increasing the girth of the code at 10. As examples of such codes are the L(m,q) codes with m>4 [11] and codes built with a geometrical construction similar to codes G_8_3 [6]. Codes with girth 10 have long code words [1], are difficult to generate, and the long codeword limits significantly the use this type of codes in mobile transmissions. The authors propose a geometrical construction of LDPC codes with d_b=3, construction which ensures a girth 8 and small number of loops with length 8 in the Tanner graph, most of the loops having lengths equaling 10 or higher values. This construction, called G_10_3, divides the check nodes in two separate groups; connections are allowed both between nodes situated in the same group, and between nodes situated in different groups. The structure oh the H matrix associated to this codes is presented in fig. 4, where A is the unitary matrix, and matrices B_{ij} have a structure also presented in fig. 4.

The B_{ij} matrices are characterized by two slopes (displacements) i and j, relatively to the A matrix, and by the difference d_{ij} between the two slopes. The construction of the H matrix implies the establishment of proper conditions for the i – j slopes and of the minimum dimension L of the elementary matrices, conditions required to suppress of the length 4 and length 6 loops and for the decrease, as much as possible, of the length 8 loops. A number of 2T slopes are generated, T being the number of B elementary matrices in the H matrix. These slopes are grouped two by two, each group being assigned to a B matrix. The slopes are generated so that a distinct difference between any two of them is ensured. For example, the pair of slopes associated to a code with rate 0.66 (H matrix composed of 6 columns with elementary matrices) are the following:

1-38 , 6–83 , 24–161 – for the upper group of check nodes ; 3–57 , 14–112 , 214–271 – for the lower group of check nodes

A - matrix

$$\begin{matrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{matrix}$$

B_{ij} - matrix

$$\begin{matrix} 0&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&1&0&1\\ 1&0&0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0&0&1\\ 1&0&1&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0&0\\ 0&0&1&0&1&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&0 \end{matrix}$$

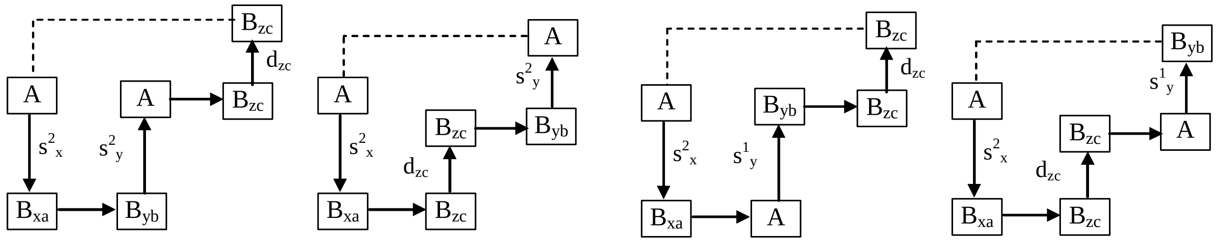
A	B_{xy}	A	B_{kt}	A
B_{ij}	A	B_{ab}	A	B_{cd}

Fig. 4 Structure of the G_10_3 codes H matrix

A. Considerations regarding the construction of the G_10_3 codes

The construction of these codes requires the suppression of loops with lengths 4 and 6 and the decrease of the number of loops with length 8, as much as possible. For the suppression of the length-6 loops, one should consider the connections between nodes from different groups and nodes from the same group. Possible connections between nodes from different groups, considering the starting point located in the A matrix, are presented in fig. 5. In this figure L stands for the dimension of the elementary matrices, s^l_x is a given slope in a B-matrix associated with group l of check nodes, $d_{ij}=s^l_i-s^l_j$ is the difference between two slopes assigned to the same B matrix. A similar situation is met when the starting point is located in a B matrix. The conditions required to suppress these loops are given in relations (1.1) – (1.4).

Note: due to the lack of space, the suppression of length 4 loops will not be discussed in this paper. Anyway, the generation algorithm of the slopes and the suppression of length-6 loops ensures the suppression of the length-4 loops.



$(s^2_x - s^2_y \pm d_{zc})(\text{mod } L) = 0$ (1.1) $(s^2_x - s^2_y \pm d_{zc})(\text{mod } L) = 0$ (1.2) $(s^2_x + s^1_y \pm d_{zc})(\text{mod } L) = 0$ (1.3) $(s^2_x + s^1_y \pm d_{zc})(\text{mod } L) = 0$ (1.4)

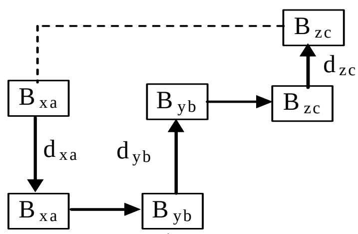
Fig. 5 Conditions required for the suppression of length-6 loops which connect check nodes from different groups.

Length-6 loops could also occur, due to connections between check nodes located in the same group. Such a situation is presented in fig. 6 and the conditions necessary to suppress these loops is expressed by relation (2)

Concerning the length-8 loops, loops that connect check nodes located in the same group and loops that connect check nodes located in different groups, could also occur. The first type of length 8 loops can be suppressed if the following relations are fulfilled:

$$(d_{xa} + d_{yb} + d_{zc} - d_{td})(\text{mod } L) = 0 \quad ; \quad (d_{xa} + d_{yb} - d_{zc} - d_{td})(\text{mod } L) = 0 \quad (3)$$

The analysis of the B-matrices structure shows that the fulfillment of the $d_{xa}=d_{zc}$ and $d_{yb}=d_{td}$ conditions, leads to length-8 loops that can not be suppressed, regardless of the elementary matrix dimension L. If the conditions (3) are fulfilled, then each check node of the Tanner graph is included in 2·Z length 8 loops, Z being the number of B matrices associated to the check node group which includes the considered node.



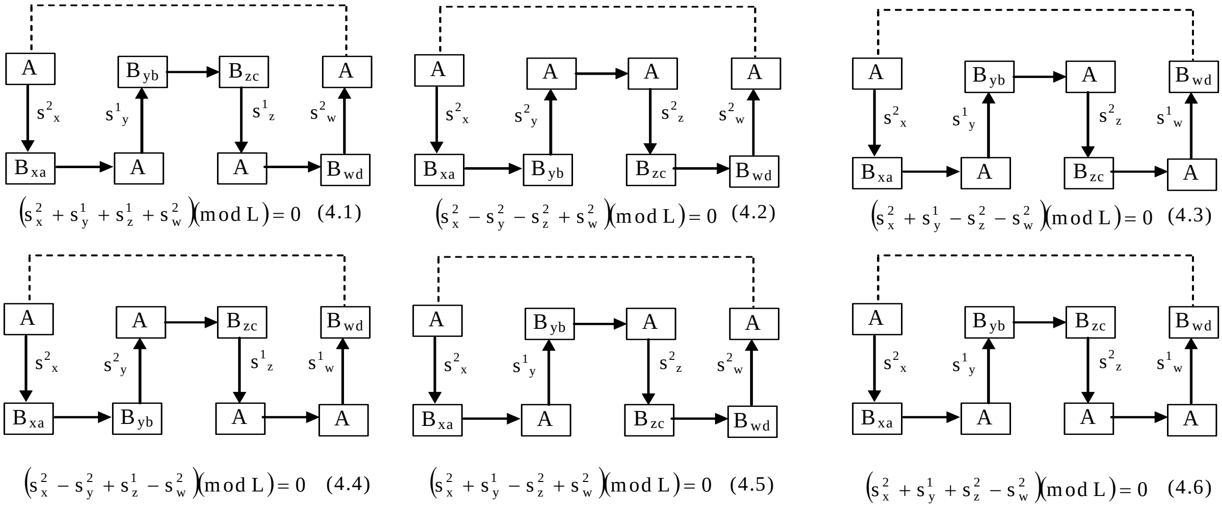
$(d_{xa} + d_{yb} \pm d_{zc})(\text{mod } L) = 0$ (2)

Fig. 6 Conditions for the suppression of length 6 loops which connect check nodes from the same group

Code/R_c	0.33	0.5	0.6	0.66
L(3,q) - L_c	27*	294	343*	512*
G_8_3 - L_c		126	216*	495
G_10_3 - L_c	234	564	1425	2856
G_10_3 – L_r	156	352	900	1656

Table 2 Minimum lengths of G_10_3, G_8_3 and L(3,q) codes

The connections between the elementary matrices which generate length-8 loops are presented in fig 7. This figure presents only the situations when the starting point is located in an A matrix, the situations when the starting point is located in a B matrix being similar. The required conditions to suppress these loops are presented in relations (4.1) – (4.6). Table 2 specifies the minimum lengths L_c of codes G_10_3, which ensure the suppression of most of the length-8 loops, for several coding rates. It also specifies the minimum lengths, L_r, required, to suppress only the length-6 loops, for the same coding rates. For comparison, this table also presents the minimum lengths L_c of the codes L(3,q) and G_8_3, for the considered coding rates.



$$\left(s_x^2 + s_y^1 + s_z^1 + s_w^2\right)(\mathrm{mod}\ L) = 0 \quad (4.1)$$

$$\left(s_x^2 - s_y^2 - s_z^2 + s_w^2\right)(\mathrm{mod}\ L) = 0 \quad (4.2)$$

$$\left(s_x^2 + s_y^1 - s_z^2 - s_w^2\right)(\mathrm{mod}\ L) = 0 \quad (4.3)$$

$$\left(s_x^2 - s_y^2 + s_z^1 - s_w^2\right)(\mathrm{mod}\ L) = 0 \quad (4.4)$$

$$\left(s_x^2 + s_y^1 - s_z^2 + s_w^2\right)(\mathrm{mod}\ L) = 0 \quad (4.5)$$

$$\left(s_x^2 + s_y^1 + s_z^2 - s_w^2\right)(\mathrm{mod}\ L) = 0 \quad (4.6)$$

Fig. 7 Conditions required for the suppression of length-8 loops which connect check nodes from different groups.

B. Performance comparison between G_10_3 and L(3,q) codes

Fig. 8 shows some comparisons between the BER/SNR performances of G_10_3 and L(3,q) codes, for different coding rates and for different code lengths, L_{code}, observing the conditions $L_{code} \geq L_c$ (fig. 8.a) and $L_c \geq L_{code} \geq L_r$ (fig. 8.b) respectively.

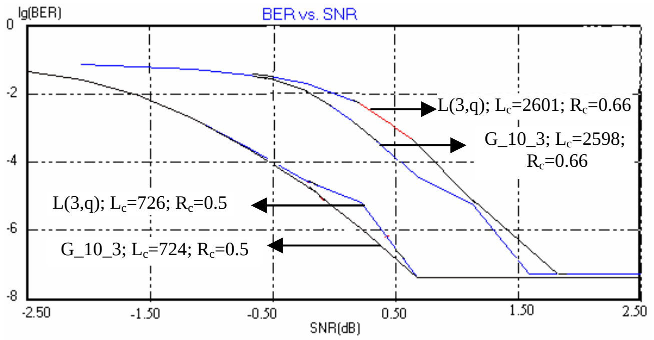


Fig. 8.a. BER vs. SNR of codes G_10_3 and L(3,q) with different L_c and R_c

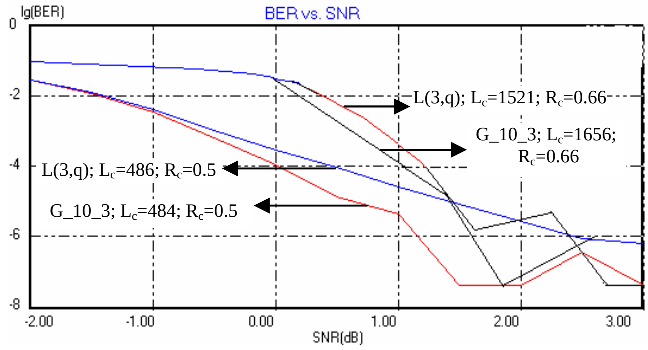


Fig. 8.b. . BER vs. SNR of codes G_10_3 and L(3,q)

The G_10_3 codes have higher minimum lengths (or even much higher) than codes G_8_3 and L(3,q), but their performances are better than those of codes G_8_3; they also ensure similar (or sometimes better) performances than the L(3,q) codes (for the same length and rate). The length of G_10_3 codes can be modified with a much smaller step than the one of the L(3,q) codes. Their code length can be increased or decreased down to the minimum length L_r, which ensures the suppression of length-6 loops; in the last case they will exhibit some performance decrease.

Note: the length of a LDPC code can be also changed by shortening, but this leads to a decreased coding rate and, possible, to a loose of performance.

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