Optimality Considerations of Short-Length LDPC Codes Construction

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Outline

1. Pseudocodewords of BP and LP decoder
2. Minimal Pseudocodewords
3. Finite Geometries Codes
4. Optimality Considerations
Finite-Length Analysis

- Asymptotic Analysis
  - Density Evolution and approximations
  - Outputs capacity-approaching degree distributions

- Finite-Length Analysis
  - Well developed in the BEC case
    - Stopping Sets
  - Pseudocodewords Theory
    - Promising direction in finite-length analysis
Pseudocodewords of Tanner Graphs

- Concept introduced by Wiberg
  - Pseudocodewords on Computation Trees
- Further Refined by Koetter and Vontobel
  - Pseudocodewords on Graph Covers
ML Decoding as a Linear Program

Easy to convert ML decoding into:

\[ \hat{x}_{ML}(y) = \hat{x}_{ML}(\lambda) = \arg \min_{x \in CH(C)} \sum_{i=1}^{n} \lambda_i \cdot x_i \]  \hspace{1cm} (1)

- CH(C): Convex Hull of set C
- Log Likelihood Ratio: \( LLR(y_i) = \lambda_i = \ln \left( \frac{Pr[y_i|x_i=0]}{Pr[y_i|x_i=1]} \right) \)
- CH(C) is of exponential description complexity
- Fails to give a practical ML decoder
Use standard Relaxation Technique from LP theory:
- Replace CH(C) with less complex polytope

Fundamental Polytope:
- Each row $[h_i]$ of $[H]$ defines local SPC code $C_i$

$$\mathcal{P}(H) = \bigcap_{i}^{m} CH(C_i)$$ (2)

If $w_H(h_i)$ is bounded by a constant as $n$ grows:
- Valid for LDPC codes
- Linear number of constraints with $n$
- Practical but suboptimal decoder
LP Decoding:

\[ \hat{x}_{LP}(y) = \hat{x}_{LP}(\lambda) = \arg\min_{x \in \mathcal{P}} \sum_{i=1}^{n} \lambda_i \cdot x_i \]  

- LP solution is always a vertex of optimization domain
- LP Pseudocodewords: set of vertices of \( \mathcal{P} \), \( V(\mathcal{P}) \)
  - \( C \subset V(\mathcal{P}) \)
  - \( F(\mathcal{P}) = V(\mathcal{P}) \setminus C \) - Set of Fractional Solutions

BP vs LP decoding

- LP decoder: Suboptimal decoder with a strong connections with BP decoder
  - Equivalent with BP decoder on BEC
  - On AWGN Channel:
    - Same characterization of the set of Pseudocodewords
    - Fundamental Polytope
  - Further references on BP and LP connections:
Conic Hull of the Fundamental Polytope $\mathcal{P}$

\[ K(H) = \{ \alpha x | \alpha \geq 0, x \in \mathcal{P} \} \]  

(4)

\[ \forall j \in \mathcal{J} : \quad \omega_j \geq 0, \]  

(5)

\[ \forall i \in \mathcal{I}, \forall j \in \mathcal{J}_i : \quad \sum_{j' \in \mathcal{J}_i \setminus \{j\}} \omega_{j'} \geq \omega_j. \]  

(6)
All-Zero Codeword LP Decoding Region

\[ D_{0}^{LP} = \{ \lambda | \lambda \cdot \mathbf{x} \geq 0, \forall \mathbf{x} \neq 0, \mathbf{x} \in V(\mathcal{P}) \} \]  \hspace{1cm} (7)

- Convex Cone with apex in the origin
- Most of the \( \mathbf{x} \in V(\mathcal{P}) \) are redundant
- Minimal Pseudocodewords \( M(\mathcal{P}) \subseteq V(\mathcal{P}) \) describes \( D_{0}^{LP} \)

\[ D_{0}^{LP} = \{ \lambda | \lambda \cdot \mathbf{x} \geq 0, \forall \mathbf{x} \neq 0, \mathbf{x} \in M(\mathcal{P}) \} \]  \hspace{1cm} (8)

- The Fundamental Cone and the All-Zero Codeword Decoding Region are Dual Cones
The Dual Cone Relationship

\[ l = x_{PC} \]

\[ \lambda x_{PC} = 0 \]
Pseudo-weight of the pseudocodewords

- Generalization of the Hamming weight
- AWGN Pseudo-weight

\[ \omega_P^{AWGN}(x) = \frac{\| x \|_1^2}{\| x \|_2^2} \]  

(9)

- Distance between plane \( \{ \gamma \in \mathbb{R}^n \mid y\gamma^\tau = 0 \} \) and point \( +1 = (1, 1, \ldots, 1) \)
- Inversely proportional to \( \angle(x, 1) \)
- We are interested in the Minimal Pseudocodeword Weight Spectrum
Finite Geometries Codes

- Experimentally observed very good performance with BP decoding

- Families of Type I PG-LDPC and Type I EG-LDPC Codes

- Parity-Check Matrix \( H \) defined by an incidence structure of points and lines in the projective plane PG(2,q)
Pseudoweight Spectrum Gap $g(H)$
- The difference between $d_{\text{min}}$ and the minimum pseudoweight over the set of fractional (non-codeword) pseudocodewords

Examine the Pseudoweight Spectrum Gap of Finite Geometries Codes

Numerical Results for Type I PG-LDPC Codes

- **PG(2,2) code**: $C_{PG(2,2)}$ [7,3,4]-code
  - $g(H_{PG(2,2)}) = 6.25 - 4 = 2.25$
- **PG(2,4) code**: $C_{PG(2,4)}$ [21,11,6]-code
  - $g(H_{PG(2,4)}) = 9.8 - 6 = 3.8$
- **PG(2,8) code**: $C_{PG(2,8)}$ [73,45,10]-code
  - $g(H_{PG(2,8)}) \approx 6$
Pseudoweight Spectrum of PG(2,4) code
Pseudoweight Spectrum of 15x15 EG(2,4) code
Pseudoweight Spectrum of 9x15 EG(2,4) code
Pseudoweight Spectrum of 8x15 EG(2,4) code
Choosing a code $\mathcal{C}$ we choose $d_{\text{min}}$
- Performance guarantees for ML decoding

Choosing the parity-check matrix $\mathbf{H}$ we choose minimum pseudoweight
- Performance guarantees for LP and BP decoding

Problem: Find families of $\mathbf{H}$ that have large Pseudoweight Spectrum Gaps

Fact: Adding redundant rows improves performance
To define \([n, k]\) binary linear code \(C\):
- \(m \times n\) parity check matrix \(H\)
- spans the space orthogonal to \(C\)
- \(m \in M = \{n - k, \ldots, 2^{n-k} - 1\}\)

\[g_C(m) = \max_{H \in \mathcal{H}_m(C)} g(H), \quad m \in M\]

\(\mathcal{H}_m(C)\): set of all \(m \times n\) parity check matrices spanning the space orthogonal to the \(C\).

\(g_C(m)\): characteristic of the code \(C\).
The Stopping Redundancy

$$\rho(C) = \min\{m \mid g_C(m) \geq 0\}$$  \hspace{1cm} (10)