On the MIMO Channel Capacity Predicted by Kronecker and Müller Models

Müge Karaman Çolakoğlu and Mehmet Şafak

***Abstract*—This paper presents a comparison between the outage capacity of MIMO channels predicted by the Kronecker and Müller models as a function of the number of scatterers, transmit and receive antennas. The Müller model is based on the single-scattered rays between arrays of transmit- and receive antennas, while the Kronecker model considers only double scattering. The channel capacity predictions by the Müller model were observed to be higher than those by the Kronecker model. Moreover, Müller model is simpler since it is characterized by fewer parameters, and accounts for frequency selective fading whilst the Kronecker model is valid only for frequency flat fading.**

***Index Terms*— Antenna arrays, communication system performance, diversity methods, eigenvalues and eigenfunctions, fading channels, information rates, MIMO systems**

I. INTRODUCTION

MIMO systems provide significant improvement in the capacity of wireless communication systems. However, an accurate model of the MIMO channel is needed for a realistic assessment of the capacity. The parameters determining the channel capacity include the number and the spacing of transmit and receive antennas and the scattering richness. Since rays undergo significant amounts of attenuation each time they are scattered, the channel capacity is also influenced by the number of times the transmitted rays are scattered in the propagation process.

This paper presents a comparative study of two models for the MIMO channels. In the Kronecker model, which applies for a flat-fading MIMO channel, the rays are assumed to be scattered twice before arriving at the receiver; thus direct and the single-scattered rays are ignored [1], [2]. In this model, fading correlations are separated at transmit and receive antenna arrays. The second model, that will be considered, is an asymptotic random channel model for a frequency-selective fading MIMO channel, where only singly scattered rays are considered [3]. This model, which will be referred to as the Müller model, is characterized by the number of scatterers, transmit- and receive antennas. Only these parameters show significant influence on the singular value distribution of the random channel matrix, and uniquely determine the channel capacity. It is simpler and emphasizes the importance of scattering richness in MIMO channels.

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M. Şafak is with the Hacettepe University, Department of Electrical and Electronics Engineering, Beytepe, 06800 Ankara, Turkey (phone: +90-312-2977070; fax: +90-312-2992125; e-mail: msafak@hacettepe.edu.tr).

M. Karaman Çolakoglu is with ASELSAN Inc., MST Division, P.O. 101, Yenimahalle, 06172 Ankara, Turkey (phone: +90-312-5922674; fax: +90-312-3545205; e-mail: mkaraman@aselsan.com.tr).

II. MIMO CHANNEL MODELS

Consider a flat-fading MIMO system with a transmit array of N_T antennas and a receive array of N_R antennas. The N_R-dimensional received signal vector **y** may be written as

$$\mathbf{y} = \mathbf{Hx} + \mathbf{n}, \qquad (1)$$

where **x** denotes the N_T-dimensional transmitted complex signal vector, and **n** is an N_R-dimensional noise vector with zero-mean independent and identically distributed (i.i.d.) complex Gaussian entries, with real and imaginary parts having equal variances. **H** denotes the N_RxN_T channel matrix with complex elements $\{h_{i,j}\}$, describing the channel gain between the *j*th transmitting antenna and the *i*th receiving antenna.

The capacity of a MIMO system when the transmitter has no channel state information (CSI) is given by

$$C = \log_2 \det\left(\mathbf{I} + \frac{\gamma}{N_T}\mathbf{HH}^H\right) \quad \left(bits/\sec/Hz\right) \qquad (2)$$

where γ is the average signal-to-noise ratio (SNR) per receive antenna, found by equally dividing the total transmit power $E[\mathbf{x}^H\mathbf{x}]$ into N_T transmit antennas, and **I** is an N_RxN_R identity matrix. The superscript $.^H$ and E[.] denote, respectively, conjugate transpose, and the mean value. Note that the capacity *C* in (2) is a random variable since the channel matrix **H** has random entries.

There are various methods for finding the average capacity. The simplest approach consists of replacing the mean value of $\mathbf{HH}^H$ in (2) by a deterministic correlation matrix to find the resulting eigenvalues and the average capacity [4],[5]. The entries of the correlation matrix may be correlated with each other depending on the antenna spacing, angular spread of the transmitted and the received signals, and the scattering richness. Since it is impossible to reduce fading correlations simultaneously between all array elements, the validity of the model chosen for the correlation matrix is critical. The second approach is based on the observation that, when the elements of **H** are zero-mean complex Gaussian random variables, $\mathbf{HH}^H$ is a central Wishart matrix. The probability density function (pdf) of the ordered eigenvalues of a complex Wishart matrix is known [6]. Ergodic and outage capacities can then be determined by using the joint pdf of the ordered non-zero eigenvalues of $\mathbf{HH}^H$ [6],[7]. In this model, it may not be easy to determine the marginal pdf's of the eigenvalues and the capacity analytically. Here, we will focus our attention on the Kronecker and the Müller models.

A. Kronecker Model

This model applies to a flat-fading channel, where the transmitter has no CSI, whilst the receiver has perfect CSI. Only the scatterers located in the vicinity of transmit and receive antenna arrays are assumed to contribute to the propagation mechanism. These scatterers are assumed to be in the far-fields of the corresponding arrays. The remote scatterers are ignored, since path losses will limit their contributions. Here, the terminology used in [1] is adopted to avoid confusion. The reader is referred to [1], [2] for a detailed description of this model.

As shown in Fig. 1, d_t and d_r denote the element spacing of a uniform linear array of N_T antennas at the transmitter and of N_R antennas at the receiver, respectively. The antenna elements are assumed to be isotropic. The S scatterers close to the transmit array (transmit scatterers) are assumed to be at least at a distance of R_{t0} from the transmit array and to be confined to a scattering radius of D_t from the line-of-sight (LOS). The exact location of the scatterers is not needed in this model. Corresponding parameters for the receiving side are R_{r0} and D_r, respectively. The number of isotropic scatterers S on both sides is assumed to be sufficiently large (typically S>10) so that random fading conditions apply. The distance between transmit and receive scatterers is denoted by R where $R \gg R_{r0}$ and R_{t0}.

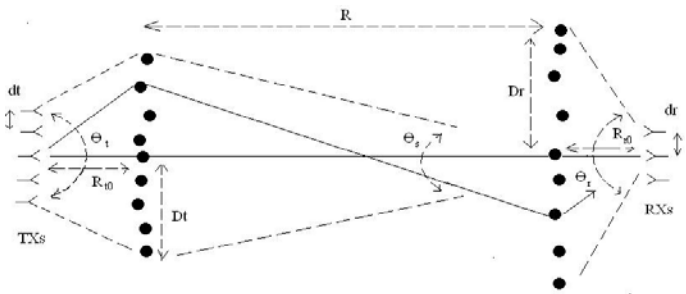

Fig. 1. Kronecker model for a fading MIMO channel

Rays transmitted by the transmit array are assumed to be scattered firstly from a transmit scatterer, with an angular spread of $\theta_t = 2\tan^{-1}(D_t/R_{t0})$, and then from a receive scatterer, with an angular spread of $\theta_r = 2\tan^{-1}(D_r/R_{r0})$, before arriving at the receive array. Receive scatterers may be considered to consist of S virtual antennas with an average spacing of $2D_r/S$, and having an angular spread of $\theta_S = 2\tan^{-1}(D_t/R)$.

The $N_R x N_T$ MIMO channel matrix is assumed to be given by [1]

$$\mathbf{H} = \frac{1}{\sqrt{S}} \mathbf{R}^{1/2}_{\theta_r,d_r}\ \mathbf{G}_r\ \mathbf{R}^{1/2}_{\theta_s,2D_r/S}\ \mathbf{G}_t\ \mathbf{R}^{1/2}_{\theta_t,d_t} \qquad (3)$$

$N_R x N_R$ matrix $\mathbf{R}_{\theta_r,d_r}$ denotes the correlation matrix governing the fading correlations between the receive antennas, while $N_T x N_T$ matrix $\mathbf{R}_{\theta_t,d_t}$ accounts for the fading correlations between the transmit antennas. The SxS matrix $\mathbf{R}_{\theta_s,2D_r/S}$ determines the fading correlations between the signals received, with an angular spread of θ_s, by the receive scatterers, separated from each other with an average spacing of $2D_r/S$. The SxN_T matrix $\mathbf{G}_t$ has i.i.d. complex Gaussian entries and accounts for the propagation between the N_T transmit antennas and the S receive scatterers. Similarly, the $N_R xS$ matrix $\mathbf{G}_r$ describes the propagation between S receive scatterers and N_R receive antennas. A factor $1/\sqrt{S}$ is used to normalize the channel energy, i.e., to make the channel energy independent of the number of scatterers.

Factorization of the channel matrix as in (3) allows the separation of the fading correlations at each stage of the propagation process. This may help the design and optimization of MIMO systems.

Kronecker model is anticipated to model better the MIMO channels in urban areas, because it ignores direct and single-scattered rays and assumes that the scattering takes place at scatterers in the vicinity of the transmitter and the receiver. This model is reported to fail in predicting the MIMO system performance under certain circumstances and to be more accurate in channels with low fading correlations [8], [9].

B. Müller Model

This is an analytical asymptotic model for a frequency selective channel. It is based on the fact that the singular values of the random channel matrix **H** shows fewer random fluctuations and eventually become deterministic as its size goes to infinity. The singular value distributions of asymptotically large matrices can be calculated analytically and only the surviving physical parameters show significant influence on the singular value distribution. Though the asymptotic distribution of singular values is only an approximation to the distribution of finite-size matrices, the asymptotic singular value distribution thus found can help identifying the dominant physical parameters which characteize a MIMO channel [3],[7],[10]. The random matrix theory is widely used in the analysis of MIMO systems. The reader is referred to [3], [11] for details.

Consider arrays of N_T transmit and N_R receive antennas in a channel with a total number of S scatterers. As shown in Fig. 2, transmit and receive antenna arrays are located at the foci of concentric ellipsoidal (equi-delay) surfaces on which scatterers are located. The scatterers located on the surface of outer ellipsoids correspond to longer delays for the received rays. The channel richness is defined as the number of scatterers that can be distinguished in delay and space coordinates. Each ray is assumed to be scattered only once before arriving at the receive antenna array, i.e., direct (LOS) ray and multiple scattering is ignored. Note that multiple scattering implies weaker received rays compared to single scattering.

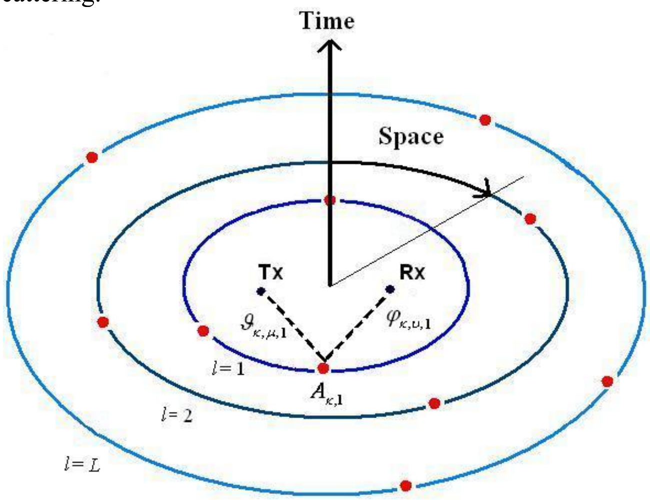

Fig. 2. Delay and space coordinates for the Müller model

Consider scattering from the surface of the *lth* ellipsoid. The propagation coefficient between μth transmit antenna and νth receive antenna is given by

$$h_{v,\mu,l} = \sum_{\kappa=1}^{S_l} A_{\kappa,l} e^{j(\vartheta_{\kappa,\mu,l}+\varphi_{\kappa,v,l})} \quad (4)$$

Here, the summation with respect to κ accounts for the contributions from all the S_l scatterers located on the surface of the *l*th ellipsoid. $\varphi_{\kappa,\upsilon,\ell}$, $\vartheta_{\kappa,\mu,\ell}$, $A_{\kappa,\ell}$ and S_ℓ show, respectively, the relative carrier phase at νth receive antenna due to κth scatterer, the relative carrier phase at κth scatterer due to μth transmit antenna, the attenuation of the κth path, and the number of scatterers at delay ℓ.

The multipath intensity profile of this frequency-selective MIMO channel is defined by the average received power from rays with Rayleigh distributed amplitudes, $A_{\kappa,l}$, corresponding to ℓth delay:

$$P_\ell \triangleq \frac{1}{S_\ell}\sum_{\kappa=1}^{S_\ell} E\left[\left|A_{\kappa,\ell}\right|^2\right], \ell = 1,2,\cdots L \quad (5)$$

N_R-dimensional received signal vector **y** at time k may be written as [3]

$$\mathbf{y}[k] = \sum_{\ell=1}^{L} \mathbf{H}_\ell \mathbf{x}[k-\ell-1], \quad (6)$$

where **x** denotes the N_T-dimensional transmitted signal vector and the N_RxN_T channel matrix $\mathbf{H}_l$ corresponds to *l*th delay. The eigenvalue distribution of $\mathbf{H}_\ell \mathbf{H}_\ell^H$, 1<*l*<L, and of $\Sigma\Sigma^H$, where $\mathbf{\Sigma} \triangleq \sum_{\ell=1}^{L} \mathbf{H}_\ell$ is the space-time channel matrix, are identical and converge to identical asymptotic limits as N_T, N_R $\to\infty$ and N_T/N_R remains fixed [3]. This implies that a channel with a multipath intensity profile (frequency-selective fading channel), defined by (5), may be considered as a channel with a single delay (flat-fading channel) if the richness per delay, S_ℓ/N_R, is replaced by the total (channel) richness, S/N_R:

$$\rho \triangleq \sum_{\ell=1}^{L} \frac{S_\ell}{N_R} = \frac{S}{N_R} \quad (7)$$

Consequently, the asymptotic eigenvalue distribution of the space-time channel matrix does not change if delay times of particular paths vary. Hence, there is no need to distinguish between the distributions of path attenuations conditioned on different delays. This implies that the multipath intensity profile, hence the frequency-selectivity of the MIMO channel does not alter the asymptotic eigenvalue distribution. To summarize, the channel is sufficiently characterized by the channel richness, $\rho=S/N_R$, the system load, $\beta=N_T/N_R$, and the distribution of attenuations A_κ, $1<\kappa<S$ as $S\to\infty$. Here, $A_\kappa=1$ is assumed for all κ values.

Note that the Kronecker model explicitly accounts for the fading correlations in the propagation process. However, the fading correlations in the channel are implicitly accounted for by the Müller model through the diversity provided by the asymptotically large channel richness and the numbers of transmit and receive antennas.

III. RESULTS

In this section, we compare the outage capacity performance of the Kronecker and the Müller models. The parameters used to characterize the Kronecker model are listed in Table I [1]. The frequency of operation was chosen to be 2 GHz.

TABLE I
PARAMETER VALUES USED FOR THE KRONECKER MODEL

d_t (m)	d_r (m)	R_{t0} (m)	R_{r0}(m)	D_t (m)	D_r (m)	R (m)
0.15	0.15	50	50	50	50	50000

Noting that $d_t=d_r=\lambda$ and angular spreads for arrival and departure, $\theta_r=\theta_t=90^0$, are sufficiently large, fading correlations between antennas in transmit and receive arrays are not very strong. The parameter values in Table I also suggest that the operation is in the far fields of both scatterers and antennas.

For a fixed value of $N_T/N_R=4$ and S=21, the outage capacity by the Kronecker model (KM) shows a steady improvement with increasing values of N_T and N_R (see Fig. 3). However, outage capacity predictions by the Müller model (MM) were observed to be relatively insensitive to the values of N_T and N_R as long as their ratio is fixed. This confirms the asymptotic character of the Müller model.

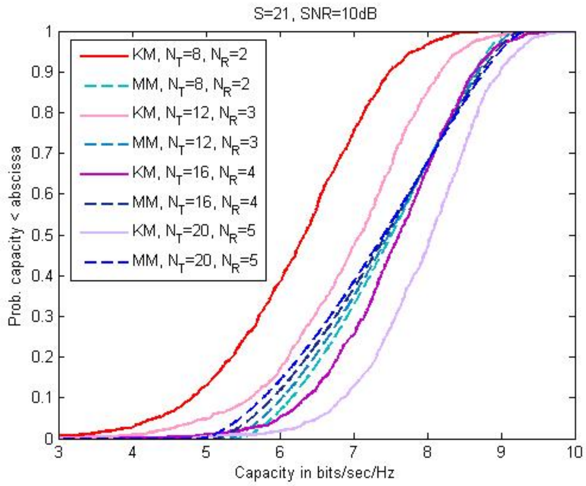

Fig. 3. The effect of the number of transmit and receive antennas on the outage capacity, predicted by Kronecker (KM) and Müller models (MM), for $\beta=N_T/N_R=4$, S=21 and SNR=10 dB.

Fig. 4 shows the effects of increase in the number of scatterers, transmit- and receive antennas on the outage capacity for $\beta=N_T/N_R=1/3$ and $\rho=S/N_R=3$. Outage capacity predictions by both models increase monotonously with increasing values of N_T, N_R and S. The predictions by the Müller model was observed to be higher than those by the Kronecker model for $N_T=1$, $N_R=3$, S=9, and for $N_T=3$, $N_R=9$, S=27 but lower for $N_T=9$, $N_R=27$, S=81. For higher values of the channel richness, $\rho=S/N_R$, the capacity was observed to improve slightly.

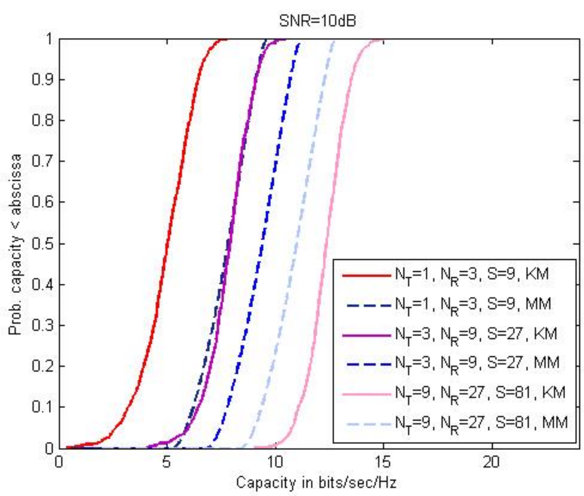

Fig. 4. The effects of N_T, N_R, and S on the outage capacity for $N_T/N_R=1/3$, $S/N_R=3$, and SNR=10 dB.

Irrespective of the value of $\beta=N_T/N_R$, an increase in N_T leads to an improvement in the Kronecker model (see Fig. 5). As for the Müller model, the outage capacity becomes higher when $N_R>N_T$. Below a threshold value of approximately 8.1 bits/sec/Hz, the outage capacity was observed to degrade with increasing values of N_T until $\beta=1$ and it begins to improve again for $\beta>1$. Above 8.1 bits/sec/Hz threshold, the outage capacity degrades monotonically with increasing values of N_T, though with increasing slopes. This can be explained by the fact that, when the channel richness and the receiver diversity are sufficiently high, transmit diversity does not provide any additional diversity advantage [12] and it may even cause a degradation in the outage capacity. In the range of capacity values below threshold, Müller model may underestimate the outage capacity compared to the Kronecker model.

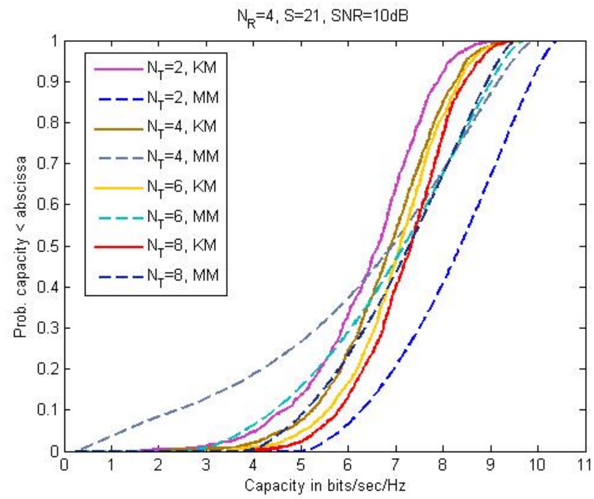

Fig. 5. The effect of N_T, the number of transmit antennas, on the outage capacity predictions by Kronecker and Müller models for $N_R=4$, S=21, and SNR=10 dB.

Fig. 6 shows the effects of the number of receive antennas on the outage capacity for $\beta=N_T/N_R<1$. In this case as well, capacity improves with N_R in both methods, though the improvement becomes less significant for higher values of N_R. Predictions by the Müller method are optimistic compared to those of the Kronecker model.

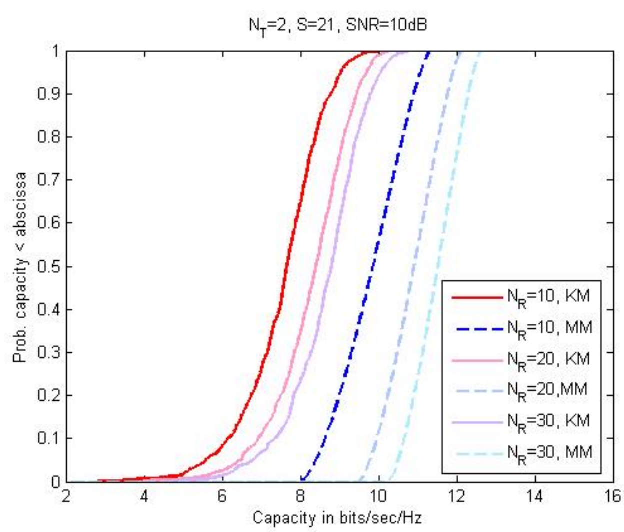

Fig. 6. The effect of N_R, the number of receive antennas, on the outage capacity predictions by Kronecker and Müller models for $\beta<1$, $N_T=2$, S=21, and SNR=10 dB.

Fig. 7 shows the effect of the channel richness on the outage capacity. The channel richness has a significant impact on the capacity predicted by both methods. Irrespective of the number of scatterers, the capacity predictions by the Kronecker model are lower than those by the Müller method. If the curves for N_R=10 and S=21 are taken as reference in Figures 6 and 7, one may easily observe that the N_R is more effective in improving the outage capacity than the channel richness for the values of the considered parameters.

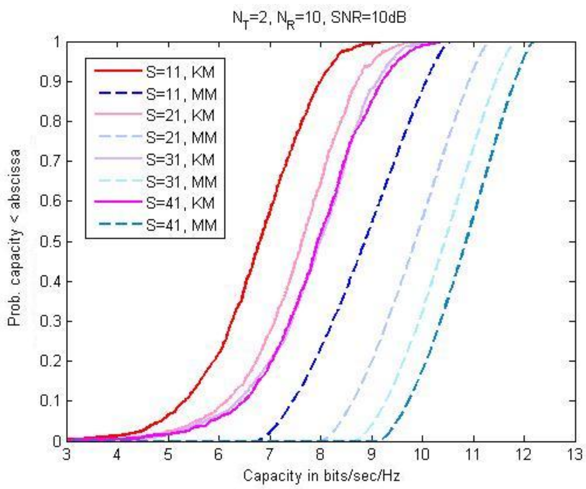

Fig. 7. The effect of the channel richness (number of scatterers) on the outage capacity predicted by Kronecker and Müller models for N_T=2, N_R=10, and SNR=10 dB.

IV. Conclusion

MIMO systems are used to improve the performance of wireless telecommunication systems. However, one needs an accurate model of the MIMO channel for this purpose. This paper presents a comparative study of the outage capacity predictions by the Müller and the Kronecker models.

The Kronecker model applies to a flat fading channel, and separates fading correlations at transmit and receive antenna arrays and in the channel. Since this model considers only the rays scattered from both transmit and receive scatterers, it may be more appropriate for radio propagation in urban areas. However, it may lead to pessimistic predictions in suburban channels, where direct- and/or singly-scattered rays may also reach the receiver. Some measurement results are reported to show that this model fails under certain circumstances.

The Müller model is simpler, valid for a frequency-selective fading MIMO channel and characterized by the scattering richness and the numbers of transmit and receive antennas.. Since it considers only the singly-scattered rays, this model may describe a suburban channel more accurately. Noting that singly-scattered rays undergo less attenuation compared with multiple-scattering, the capacity predictions by the Müller model may be higher compared with the Kronecker model. It may even overestimate the capacity in urban channels where rays undergo mostly multiple-scattering. In this model, the scattering richness plays an important role, which is usually ignored in other models.

Additional parameters needed for characterizing the Kronecker model are assumed fixed for comparison purposes. The two models are then compared as a function of the number of scatterers, transmit- and receive antennas and the SNR per receive antenna. The capacity predictions by both methods were observed to improve with increasing values of the number of scatterers.

An increase in the number of receive antennas leads to an improvement in the outage capacity predictede by both methods but the rate of improvement tapered with increasing values of the number of receive antennas.

The Kronecker model shows a steady improvement in the outage capacity as the transmit array size increases, while the Müller model behaves differently for different values of the outage capacity. When the transmit array size is higher than the receive array size, the channel capacity, predicted by the Müller model, was observed to increase with the increasing number of transmit antennas for low capacity values, but to decrease for higher capacity values. When the transmit array size is less than the receive array size, the outage capacity was observed to decrease with increasing transmit array size.

These results need to be supported by measurements in realistic scenarios.

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