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Performance Comparison of LDPC Codes Generated With Various Code-Construction Methods

The paper focuses on the following questions:

- Comparison between the BER vs. SNR performances of regular LDPC codes with the same rate, length and girth, generated by several construction methods; the girth-values considered are g = 6, 8, 12.

- Presentation and analysis of a LDPC code construction algorithm with g = 8, and with a small number of length-8 loops (girth almost 10) developed by the authors; a performance comparison between these codes and the codes generated by some other code construction methods.

1. Introduction

High throughput and/or low bit error probability in mobile transmissions

↓

- adaptive change of the QAM-constellation and correction coding rate
- channel state prediction

+

low latency transmission

↓

codeword length rather small (usually smaller than 1000)

LDPC codes → - high coding gains ensured with rather high coding rates

- relatively small complexity decoding algorithm

↓

LDPC codes are significant candidates for such transmissions

The performances of the LDPC codes, decoded with the Sum-Product algorithm [2], depend mostly on the following parameters:

- **Codeword length** – long codeword codes (thousands of bits) ensure better performances than short codeword codes (hundreds of bits) [2].
- **The number of bits of „1" in the columns of the control matrix H** – the order of the bit-nodes $d_{b.}$
- **The number of bits of „1" in the rows of H** – the order of the check-nodes d_c [4].
- **The effective distribution of „1" bits** on the columns and rows of H check matrix.
- **The minimum-length loop of the bit-nodes (girth)** within the Tanner graph associated to the code [6] [7] ; higher girth → better performances
- **The minimum Hamming distance of the code** – has a small influence when the codes are decoded with the Sum-Product algorithm.

The girth and the coding rate impose the minimum codeword length.

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2. Analyzed code construction methods

L(m,q) Codes [11]

- codeword length $N = q^m$, where q is a prime number or a power of a prime number and m is a natural number.
  - due to limitation of the codeword length, only codes with m = 2 or 3 were considered.
- the control matrix H of these codes is generated by removing a number of rows, according to the coding rate desired, out of a square matrix M ($q^m \times q^m$) ;
  - matrix M is composed of square sub-matrices of ($q^{m-1} \times q^{m-1}$) elements each, sub-matrices obtained by permutations from a basis sub-matrix.
- to generate a regular code with a coding rate $R_c = 1 - k/q$, only $k \cdot q^{m-1}$ rows should be retained from the complete matrix;
  - k should equal 3 if a bit-node order $d_b = 3$ is to be ensured.

A_1	A_2	A_3	……	A_k
B_1	B_2	B_3	……	B_k
C_1	C_2	C_3	……	C_k

Fig. 1 Structure of the L(2,q) and L(3,q) codes H matrices for d_b=3

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Codes Generated by Combinatorial Constructions

- The most studied constructions of this type [3] [4] [8] are based on the Kirkman triple systems [15], a particular case of the Steiner triple systems [15].
- These systems allow the generation of codes with $d_b = 3$, constant order of d_c and girth = 6.
- The generated codes have rather short code words, for high coding rates.
- The H control matrix of these codes is the incidence matrix of the system [4].

Codes Generated by Geometrical Constructions

- The geometrical constructions employ the structure-graph associated to the code [5] – it describes only the connections between the check-nodes.
- By splitting the check nodes into several groups and imposing restrictions for the connections between the elements of different groups, codes with a certain girth can be generated.
- Due to the codeword length limitations, only the following codes were considered:
  - codes with g = 12 and $d_b = 2$, denoted by G_12_2.
  - codes with g = 8 and $d_b = 3$, denoted by G_8_3.

Several codes considered

R_c → Code ↓	0.33	0.5	0.6	0.66	0.7	0.75	0.777	0.8	0.81	0.84	0.87	0.89	0.93	0.94
L(2,q) / L_c	20*	16	49*		81*	121*	169*		256*	361*	529*	729	1848*	2208*
L(3,q) / L_c	27*	294	343*	512*	1331*	1331	2197*							
Kirkman / L_c					70					249		532	1426	2035*
G_12_2 / L_c	39	60	145	282	469*	760	1125	1670*	2453*					
G_8_3 / L_c		126	216*	495	570	756	949*	1185	1296*	2970*				

Table 1 Minimum code lengths of LDPC codes generated with the studied code construction methods for some imposed rates

* these codes correspond only approximately to the mentioned coding rates

3. BER vs. SNR Performance Comparison between the LDPC Codes Generated With The Studied Construction Methods

- codes constructed with different methods, but having the same girth and about the same coding rate and length, were evaluated using computer simulations

Simulation conditions:

- 2PSK modulation
- AWGN channel
- Sum-Product decoding algorithm with 100 iterations/codeword

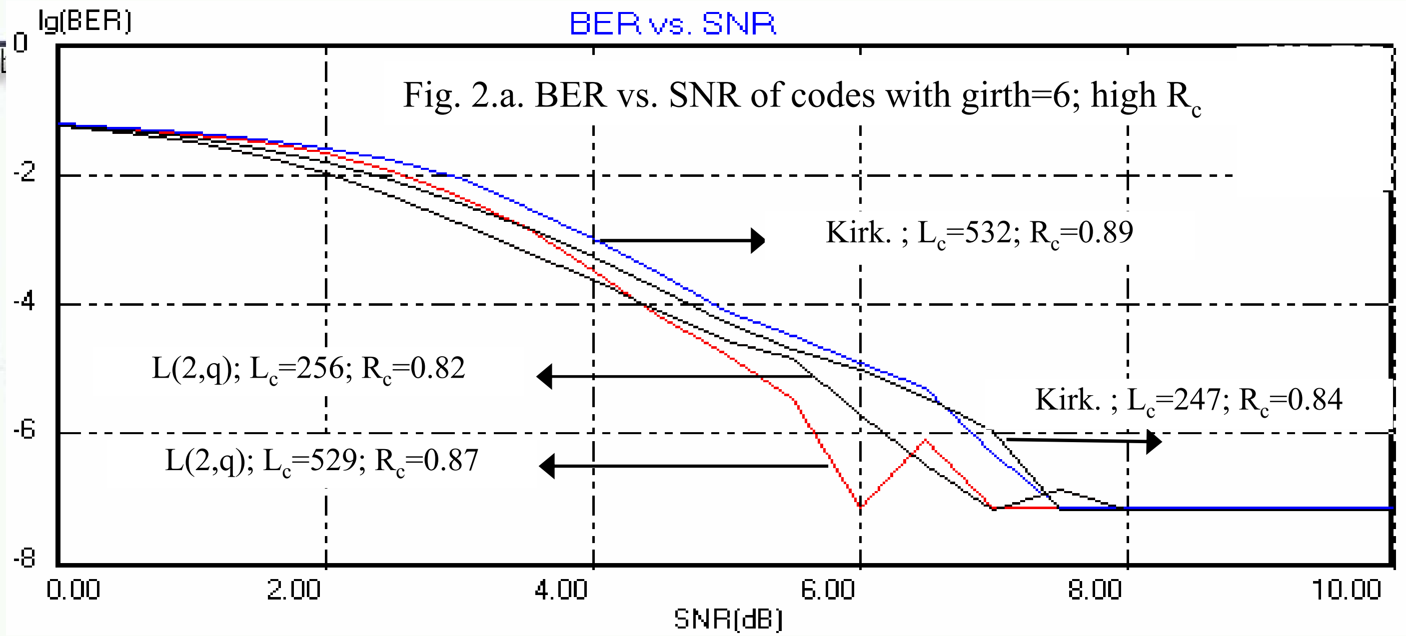
- The girth was computed using an algorithm implemented by the authors.

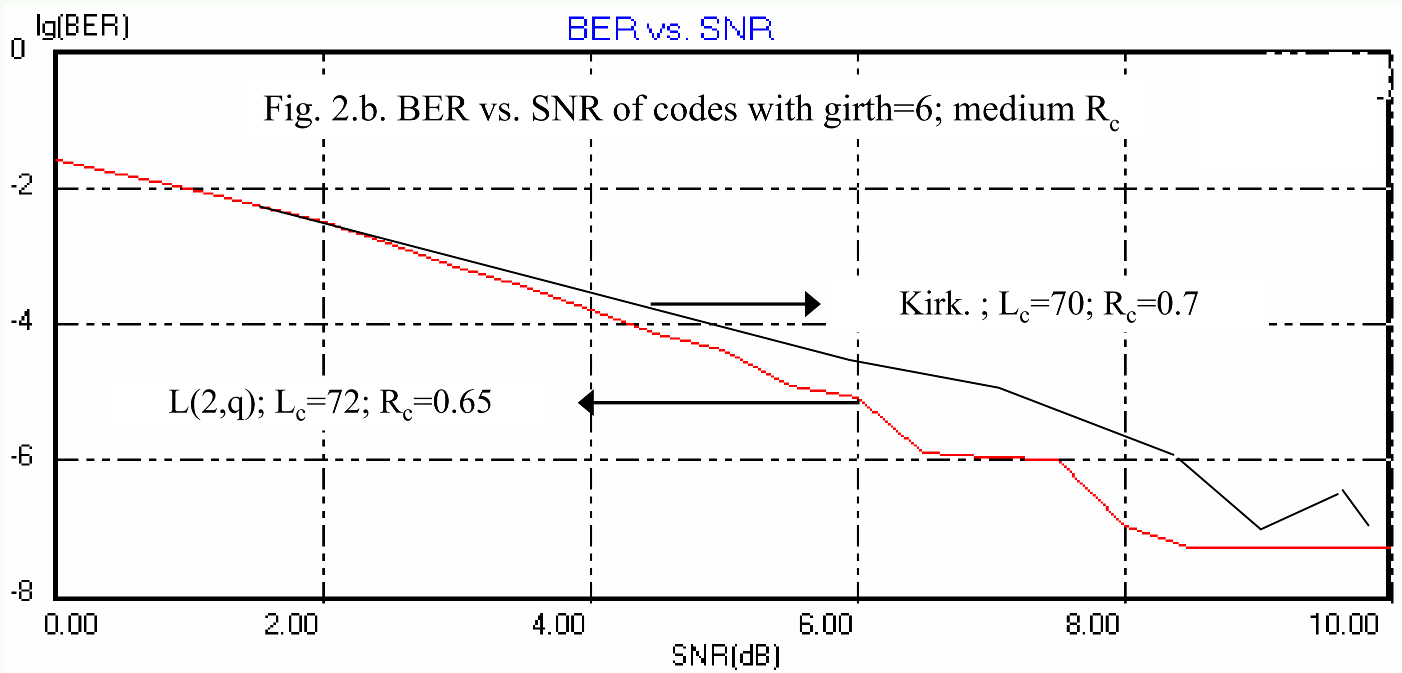
Performance Comparison of LDPC Codes With Girth = 6

BER(11.5 dB) =

= 10^{-7} - 2-PSK

non-coded





- L(2,q) codes ensure better performances than the Kirkman codes
  - the difference is about 1 – 1.5 dB, for low bit error rates, for the same coding rate and code length.
- there should be noticed the influence of the codeword length upon the coding gain:
  - see the $R_c = 0.7$ and $L_c = 70$ codes of fig.2.b, compared to the $R_c = 0.89$ and significantly longer codes $L_c = 532$ of figure 2.a.
  - the coding gain raises from 3 dB (L_c=70) to 4.5 dB (L_c=532).
- The Kirkman codes ensures high coding rates for rather small length codeword ($R_c = 0.89$ for $L_c = 532$ bits), but the coding gains are not very high, 4.5 dB (still comparable to the K=3, 4 $R_c = 1/2$ convolutional codes).

Performance Comparison of LDPC Codes With Girth Equaling 8 or 12

BER(11.5 dB) =

= 10^{-7} - 2-PSK

non-coded

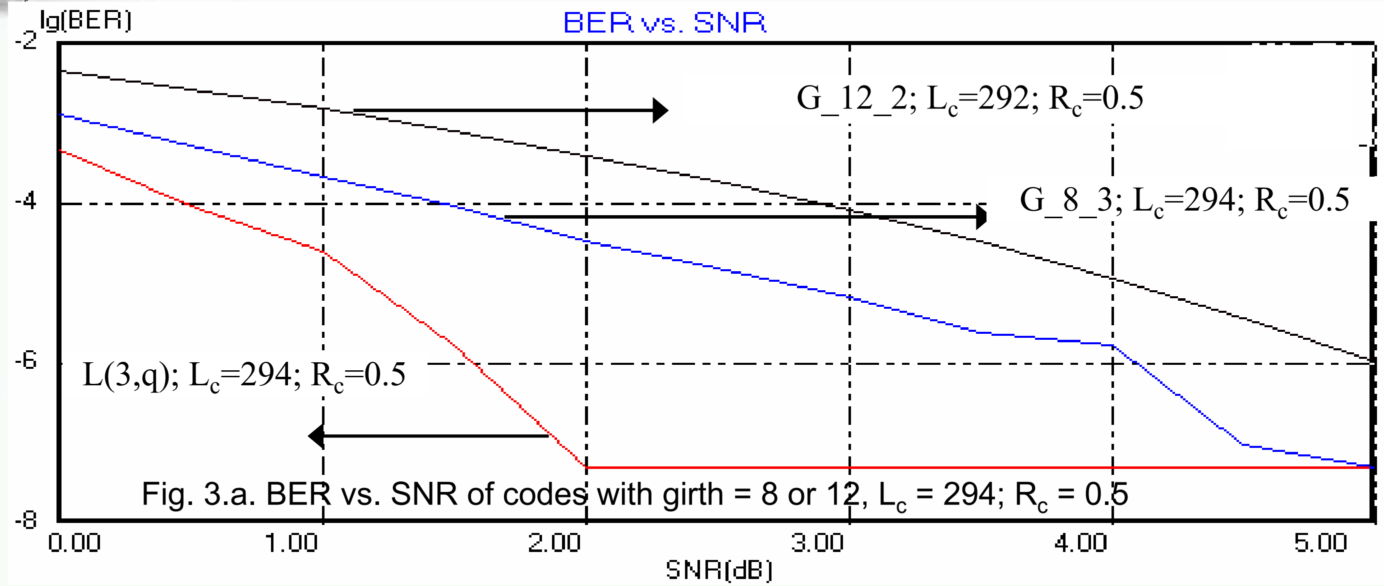


Fig. 3.a. BER vs. SNR of codes with girth = 8 or 12, L_c = 294; R_c = 0.5

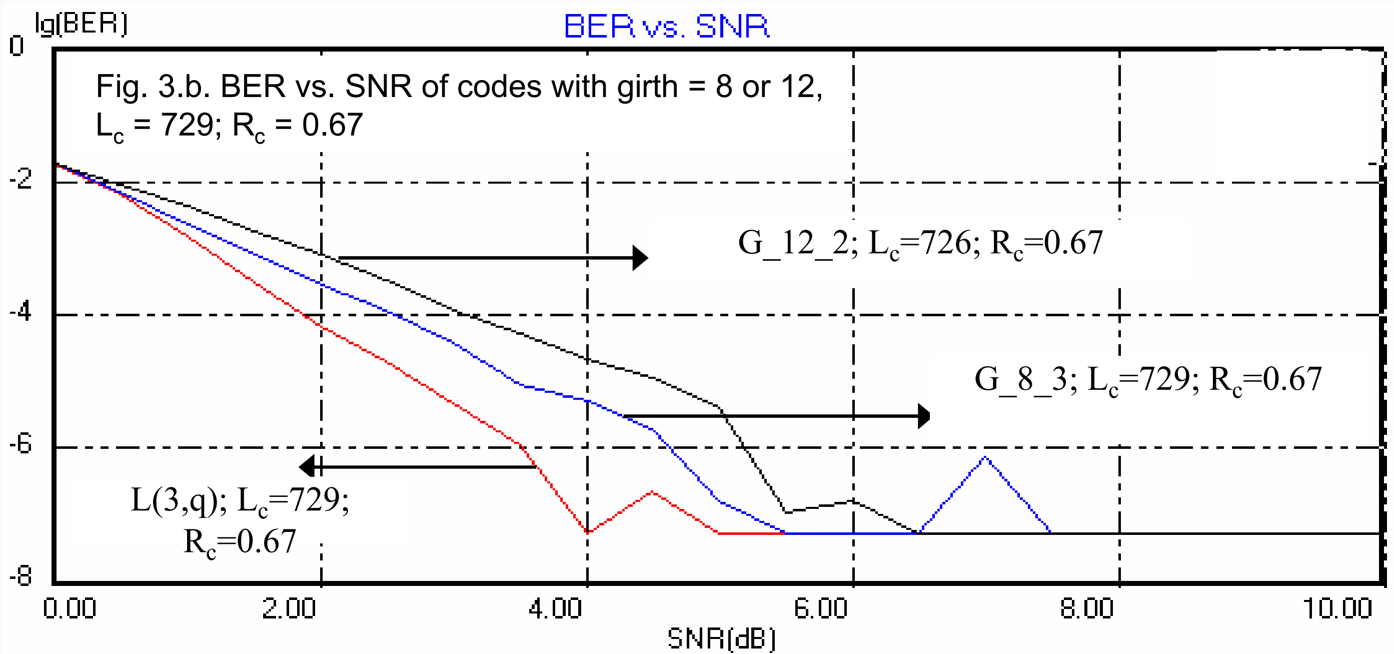
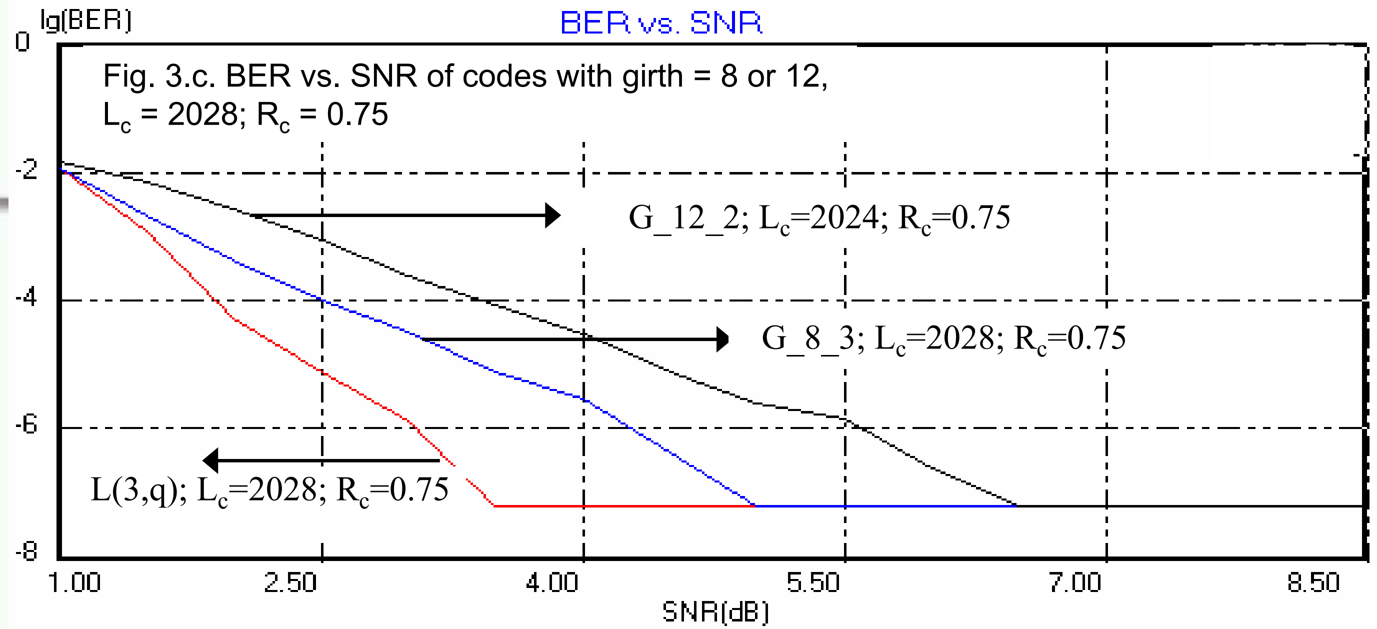


Fig. 3.b. BER vs. SNR of codes with girth = 8 or 12, L_c = 729; R_c = 0.67

BER(11.5 dB) =

= 10^{-7} - 2-PSK

non-coded



- for long code words, the L(3,q) codes have better performances than the G_8_3 codes with about 1 – 1.5 dB at low BER, for an imposed rate and codeword length;
- for short code words, the L(3,q) codes have better performances than the G_8_3 codes with about 2 - 2.5 dB at low BER, for an imposed rate and codeword length;
- the G_8_3 code have better performances than the G_12_2 codes, with about 1.5 dB at low BERs ; $d_b = 2$ for G_12_2 codes and $d_b = 3$ for G_8_3 codes.
- the minimum length of G_8_3 codes for a given rate is smaller than the length of L(3,q) codes, and this length can be changed more easily.
- the structure of the G_8_3 codes check matrix H is that presented in fig.1, the A_i matrices being unitary ;
- the dimension of the elementary matrices is lower bounded – the only restriction.

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4. Codes With Girth Almost 10

- better performances than those of L(3,q) codes (with girth 8) can be obtained by increasing the girth of the code at 10.
- girth 10 codes are the L(m,q) codes with m>4 [11] and codes built with a geometrical construction similar to codes G_8_3 [6].
- codes with girth 10 have long code words [1], are difficult to generate, and the long codeword limits significantly the use this type of codes in mobile transmissions.
- a geometrical construction of LDPC codes with d_b=3, girth 8 and small number of loops with length 8 in the Tanner graph is proposed - G_10_3 codes.
- Structure of the G_10_3 codes:
  - the check nodes are divided in two separate groups.
  - connections are allowed both between nodes situated in the same group, and between nodes situated in different groups.
  - the H matrix is composed of two type of elementary matrices (fig. 4):
    - A is the unitary matrix.
    - B_{ij} matrices are characterized by two slopes (displacements) i and j, relatively to the A matrix, and by the difference d_{ij} between the two slopes.

Considerations regarding the construction of the G_10_3 codes

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A - matrix

1	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	1

j i

B_{ij} - matrix

0	0	0	0	0	1	0	1	0	0
0	0	0	0	0	0	1	0	1	0
0	0	0	0	0	0	0	1	0	1
1	0	0	0	0	0	0	0	1	0
0	1	0	0	0	0	0	0	0	1
1	0	1	0	0	0	0	0	0	0
0	1	0	1	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0
0	0	0	1	0	1	0	0	0	0

A	B_{xy}	A	B_{kt}	A	B_{ef}
B_{ij}	A	B_{ab}	A	B_{cd}	A

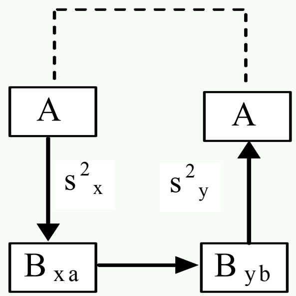
Fig. 4 Structure of the G_10_3 codes H matrix

- construction of the H matrix = finding the i – j slopes and the minimum dimension L of the elementary matrices, which ensure the suppression of the length-4 and length-6 loops and the decrease of the number of length-8 loops.
- 2T slopes are generated, T being the number of B elementary matrices in the H matrix.
  - the slopes are grouped two by two, each group being assigned to a B matrix.
  - the slopes are generated so that a distinct difference between any two of them is ensured.
  - example: the pair of slopes associated to a code with rate 0.66 (H matrix composed of 6 columns with elementary matrices – see fig. 4) are the following:
    - 1 – 38 (x,y) , 6 - 83 (k,t), 24 - 161 (e,f) – for the upper group of check nodes.
    - 3 - 57 (i,j), 14 - 112 (a,b), 214 – 271 (c,d) – for the lower group of check nodes.

Conditions necessary for the suppression of length 4 and length 6 loops

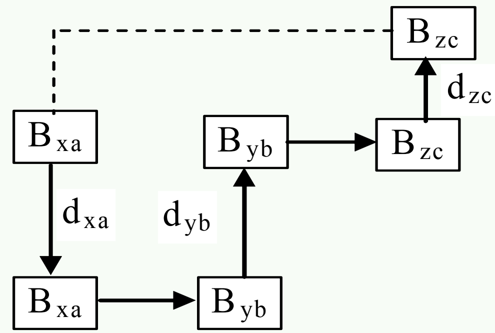
- Possible connections between nodes from different groups, considering the starting point located in the A matrix, are presented in fig. 5 – length-4 loops and fig.6 – length-6 loops

- Possible connections between nodes from the same group are presented in fig. 7 for length-6 loops



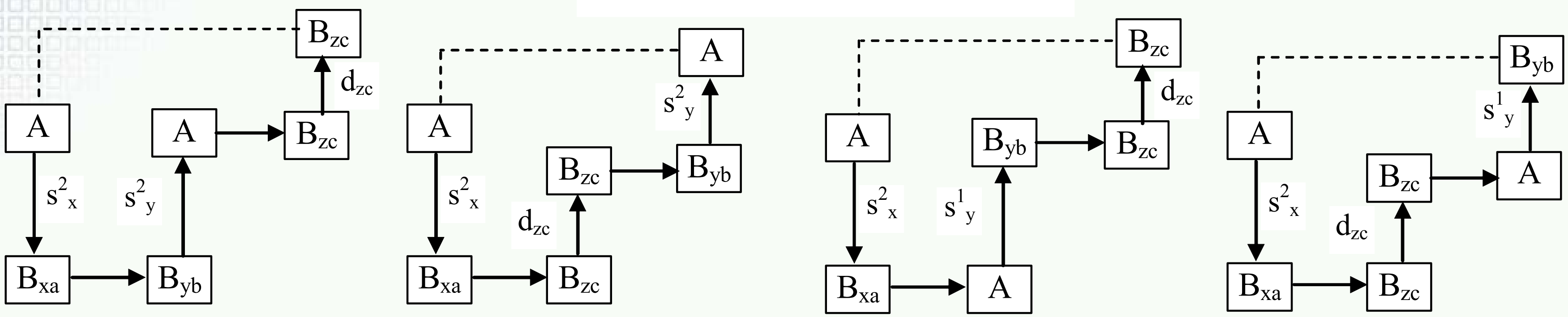
$$\left(s^2_x - s^2_y\right)(\mathrm{mod}\ L) = 0 \quad (1)$$

Fig. 5 Length-4 loops which connect check nodes from different groups and relations which describe these loops.



$$\left(d_{xa} + d_{yb} \pm d_{zc}\right)(\mathrm{mod}\ L) = 0 \quad (3)$$

Fig. 7 Length-6 loops which connect check nodes from the same group and relations which describe these loops.



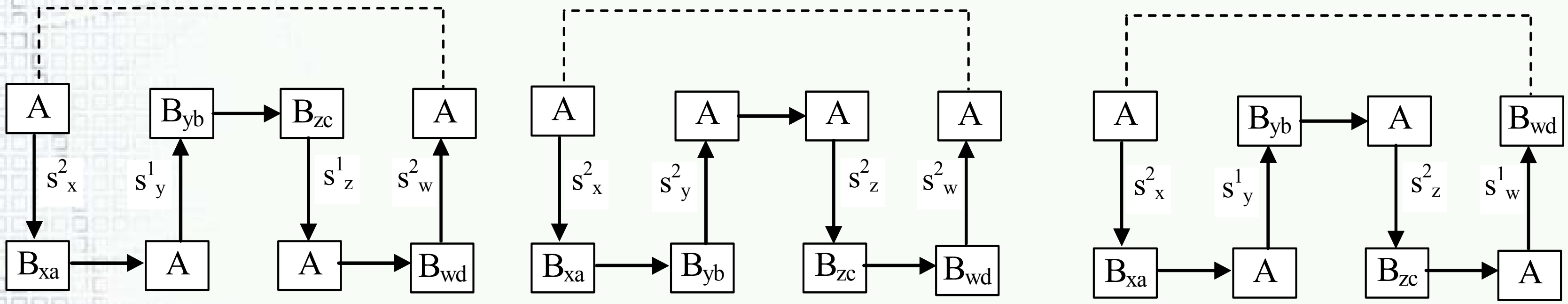
$$\left(s^2_x - s^2_y \pm d_{zc}\right)(\mathrm{mod}\ L) = 0 \quad (2.1) \qquad \left(s^2_x - s^2_y \pm d_{zc}\right)(\mathrm{mod}\ L) = 0 \quad (2.2)$$

$$\left(s^2_x + s^1_y \pm d_{zc}\right)(\mathrm{mod}\ L) = 0 \quad (2.3) \qquad \left(s^2_x + s^1_y \pm d_{zc}\right)(\mathrm{mod}\ L) = 0 \quad (2.4)$$

Fig. 6 Length-6 loops which connect check nodes from different groups and relations which describe these loops.

Conditions necessary for the suppression of length 8 loops

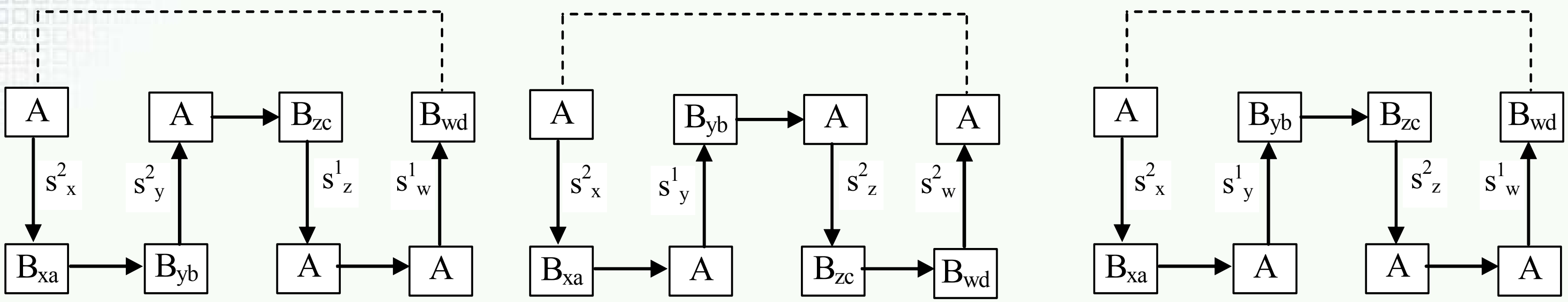
- connections between the elementary matrices which generate length-8 loops are presented in fig 8.
  - only the situations when the starting point is located in an A matrix are considered; the situations when the starting point is located in a B matrix are similar.



$$\left(s_x^2+s_y^1+s_z^1+s_w^2\right)(\mathrm{mod}\,L)=0 \quad (4.1)$$

$$\left(s_x^2-s_y^2-s_z^2+s_w^2\right)(\mathrm{mod}\,L)=0 \quad (4.2)$$

$$\left(s_x^2+s_y^1-s_z^2-s_w^2\right)(\mathrm{mod}\,L)=0 \quad (4.3)$$



$$\left(s_x^2-s_y^2+s_z^1-s_w^2\right)(\mathrm{mod}\,L)=0 \quad (4.4)$$

$$\left(s_x^2+s_y^1-s_z^2+s_w^2\right)(\mathrm{mod}\,L)=0 \quad (4.5)$$

$$\left(s_x^2+s_y^1+s_z^2-s_w^2\right)(\mathrm{mod}\,L)=0 \quad (4.6)$$

Fig. 8 Length-8 loops which connect check nodes from different groups and relations which describe these loops.

- length-8 loops that connect check nodes located in the same group can be suppressed if the following relations are not fulfilled:

$$(d_{xa} + d_{yb} + d_{zc} - d_{td})(\mathrm{mod}L) = 0 \quad ; \quad (d_{xa} + d_{yb} - d_{zc} - d_{td})(\mathrm{mod}L) = 0 \quad (5)$$

- the analysis of the B-matrices structure shows that the fulfillment of the $d_{xa}=d_{zc}$ and $d_{yb}=d_{td}$ conditions, leads to length-8 loops that can not be suppressed, regardless of the elementary matrix dimension L.

- if conditions (5) are fulfilled, then each check node of the Tanner graph is included in 4·Z length-8 loops, Z being the number of B matrices associated to the check node group which includes the considered node.
  - an example of such a situation is presented in fig. 9

B_{13} - matrix

0	0	0	0	0	0	1	0	1	0
1	0	0	0	0	0	0	1	0	1
0	1	0	0	0	0	0	0	1	0
1	0	1	0	0	0	0	0	0	1
0	1	0	1	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0
0	0	0	1	0	1	0	0	0	0
0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	1	0	1	0	0
0	0	0	0	0	0	1	0	1	0

B_{47} - matrix

0	0	0	1	0	0	1	0	0	0
0	0	0	0	1	0	0	1	0	0
0	0	0	0	0	1	0	0	1	0
0	0	0	0	0	0	1	0	0	1
1	0	0	0	0	0	0	1	0	0
0	1	0	0	0	0	0	0	1	0
0	0	1	0	0	0	0	0	0	1
1	0	0	1	0	0	0	0	0	0
0	1	0	0	1	0	0	0	0	0
0	0	1	0	0	1	0	0	0	0

Fig. 9 Example of length-8 loops which can not be suppressed

Performance comparison of G_10_3 and L(3,q) codes

Code/R_c	0.33	0.5	0.6	0.66
L(3,q) / L_c	27*	294	343*	512*
G_8_3 / L_c		126	216*	495
G_10_3 / L_c	234	564	1425	2856
G_10_3 / L_r	156	352	900	1656

Table 2 Minimum lengths of G_10_3, G_8_3 and L(3,q) codes

- minimum lengths L_c of codes G_10_3 ensure the suppression of most of the length-8 loops.
- minimum lengths L_r of codes G_10_3 ensure the suppression of length-6 loops.

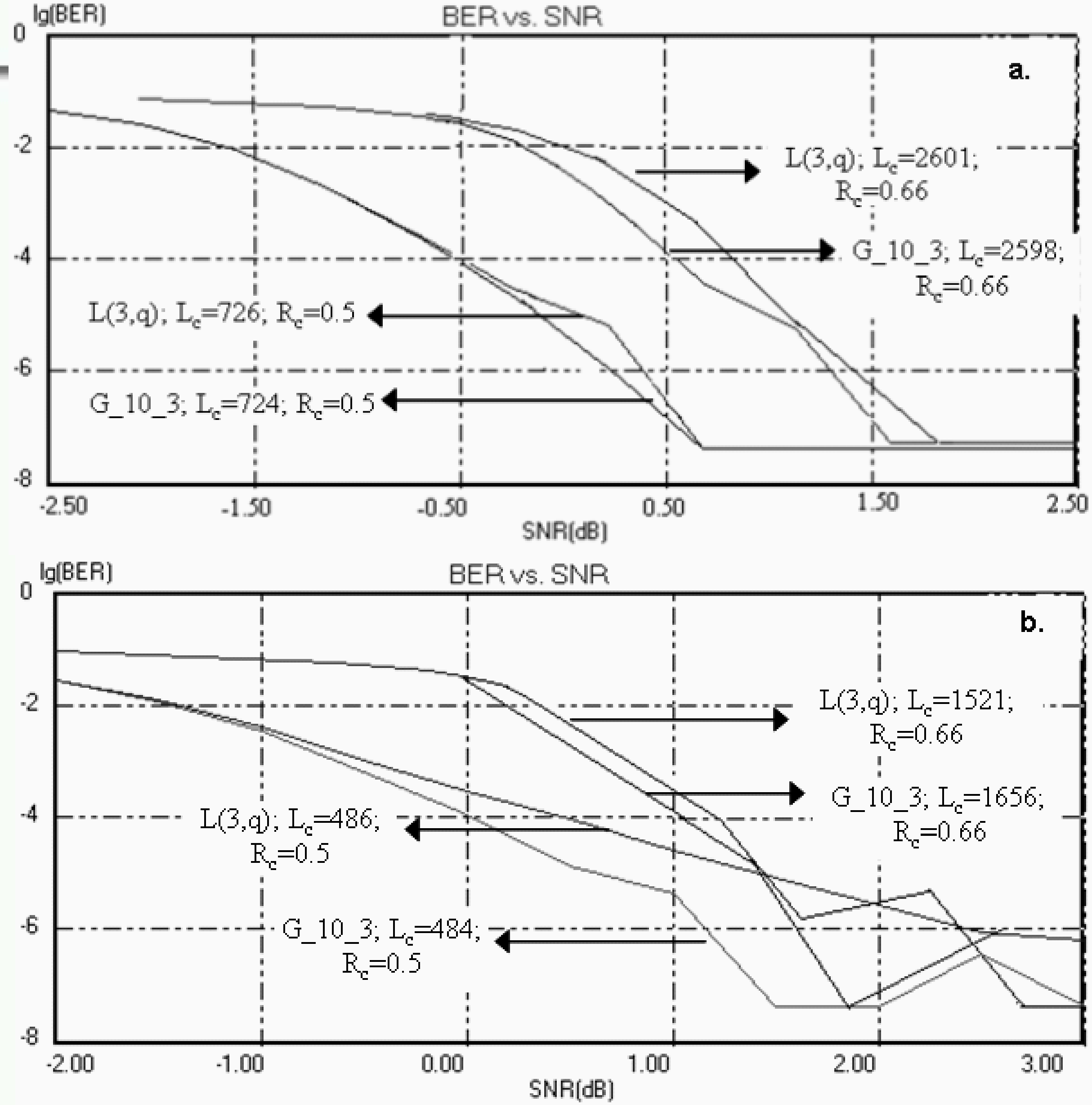


Fig. 10.a., b. BER vs. SNR of codes G_10_3 and L(3,q) with different code length L_c and coding rate R_c

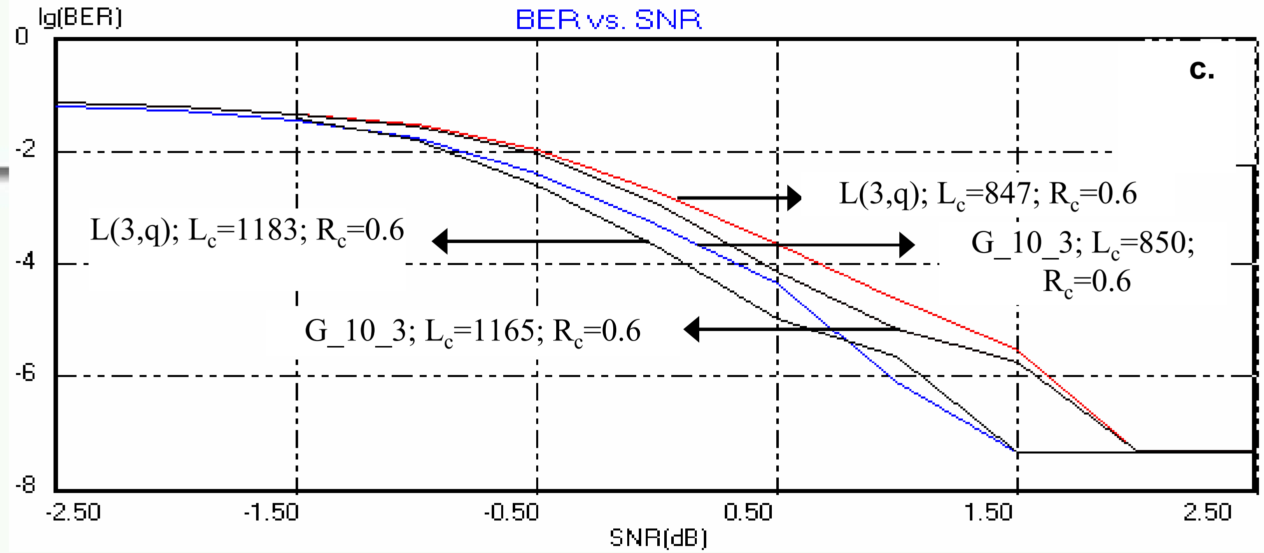


Fig. 10. c BER vs. SNR of codes G_10_3 and L(3,q) with different code length L_c and coding rate R_c

- the G_10_3 codes have higher minimum lengths (or even much higher) than codes G_8_3 and L(3,q).
- G_10_3 codes performances are better than those of codes G_8_3.
- G_10_3 codes ensure performances similar to (sometimes better, sometimes worse) L(3,q) codes (for the same length and rate).
- the length of G_10_3 codes can be modified with a much smaller step than the length of L(3,q) codes.
- the G_10_3 codes length can be increased or decreased down to the minimum length L_r, which ensures the suppression of length-6 loops; in the last case the G_10_3 codes will exhibit some performance decrease.
- G_10_3 codes with odd number of elementary matrix columns have different d_c parameter for check nodes included in different groups (fig.4).
- Note: the length of a LDPC code can be also changed by shortening, but this leads to a decreased coding rate and, possible, to a loose of performance.

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