Array Relay

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Abstract— **Relaying offers a diversity path to wireless mobile users to improve the performance when the direct link between the source and the destination is a poor scattering channel. In this paper, we present the performance of a relay, consisting of an array of antennas. While the source and the destination employ single antennas, the relay consists of a uniform linear array (ULA). The relay simply multiplies the incoming vector with a square matrix and forwards the resulting vector to the destination. The relay matrix is optimized so that the minimum mean square error (MMSE) between the symbol transmitted by the source and the symbol received at the destination is minimized. The simulations show that the maximum spectral efficiency attainable increases with the number of elements in the ULA and it is maximized if the SNR at the input of the relay and the SNR at the input of the destination is made to be equal.**

I. INTRODUCTION

Wireless communication systems suffer from small scale fading, which causes random fluctuations in the received signal. The performance degradation due to fading can be alleviated by employing diversity techniques, such as time, frequency, or space diversity. The space diversity is generally studied from various perspectives: Firstly, multiple antennas are employed at the source node and/or at the destination node and the system performance is optimized considering some criteria, e.g., maximum diversity gain, maximum multiplexing gain or a reasonable compromise between them [1]. The diversity may also be obtained by using virtual antenna arrays, e.g., using the antenna of a cooperating user in order to obtain a diversity path. This is known as user cooperation diversity [2]-[4]. Similarly, dedicated relay elements may be used to get diversity gain [5]. In cases where it is not possible to place more than one antenna to mobile nodes, the user cooperation diversity or relaying may offer diversity advantage. The special case of amplify-and-forward (AF) relaying has been investigated for multiple-hop links [6]. The smart antennas are also used for adaptive beamforming, which provides digital adaptive steering [7]. It is well known that the use of an array of a large number of closely spaced antennas, combined with adaptive beamforming methods provides large array gains, and thus improves the system performance.

In the AF relaying, relays consisting of single antennas are considered, but, to the best knowledge of the authors, the use of array antennas as relays has not been yet considered. In this paper, the use of arrays in AF relaying will be investigated as a means to achieve high spectral efficiencies. The use of an antenna array in the relay station may provide the advantages of smart antenna technology, while preserving the single antennas in mobile users.

II. PROBLEM FORMULATION

A. System Model

In the first time slot, the source node transmits the symbol x to the relay. The symbol x has an average energy $E_S = E\left[|x|^2\right]$ where $E[.]$ denotes the expectation operator. The relay receives the vector signal through its N antennas as

$$\mathbf{y}_R = \mathbf{\Phi} x + \mathbf{n}_R \tag{1}$$

where

$\mathbf{\Phi}$ is $N \times 1$ vector of complex channel coefficients between the source and the relay, whose ith element represents the channel coefficient from the source to the ith antenna element of the array.

$\mathbf{n}_R$ is $N \times 1$ complex vector of array relay noise whose ith element represents the thermal noise for the ith antenna channel of the array. We assume that $E[\mathbf{n}_R] = \mathbf{0}$ and $E\left[\mathbf{n}_R \mathbf{n}_R^H\right] = \sigma_R^2 \mathbf{I}$.

The relay multiplies the incoming signal vector by a $N \times N$ matrix $\mathbf{R}$ (not necessarily diagonal) with complex valued entries. The signal vector to be forwarded to the destination node is given by

$$\mathbf{y}_T = \mathbf{R}\mathbf{y}_R \tag{2}$$

where the average output energy of the relay is constrained to be equal to $E_R = E\left[||\mathbf{y}_T||^2\right]$.

In the second time slot, the relay transmits $\mathbf{y}_T$ to the destination. The signal received by the destination is given by

$$z = \boldsymbol{\theta}^H \mathbf{y}_T + n_D \tag{3}$$

where

$\boldsymbol{\theta}^H$ is $1 \times N$ row vector of complex channel coefficients between the relay and the destination, whose

ith element represents the channel coefficient from the ith antenna element of the array to the destination.
n_D denotes the complex, thermal noise at the destination. We assume that $E\left[n_D\right] = 0$ and $E\left[\left|n_D\right|^2\right] = \sigma_D^2$.

If (1), (2) and (3) are combined, the end-to-end system equation is obtained as

$$z = \boldsymbol{\theta}^H \mathbf{R} \boldsymbol{\Phi} x + \boldsymbol{\theta}^H \mathbf{R} \mathbf{n}_R + n_D \tag{4}$$

where the first term denotes the signal component while the last two terms represent the noise component.

B. Design of the Relay Matrix

1) Design Criteria: The relay matrix $\mathbf{R}$ will be selected such that the MMSE between the transmitted symbol and the received signal:

$$\begin{aligned} E\left[\left|z - x\right|^2\right] &= E_S \boldsymbol{\theta}^H \mathbf{R} \boldsymbol{\Phi} \boldsymbol{\Phi}^H \mathbf{R}^H \boldsymbol{\theta} \\ &\quad - E_S \boldsymbol{\theta}^H \mathbf{R} \boldsymbol{\Phi} - E_S \boldsymbol{\Phi}^H \mathbf{R}^H \boldsymbol{\theta} + E_S \\ &\quad + \sigma_R^2 \boldsymbol{\theta}^H \mathbf{R} \mathbf{R}^H \boldsymbol{\theta} + \ \sigma_D^2 \end{aligned} \tag{5}$$

will be minimized subject to the constraint that the average energy of the relay output (2) will be E_R:

$$E\left[\left\|\mathbf{y}_T\right\|^2\right] = E_S \boldsymbol{\Phi}^H \mathbf{R}^H \mathbf{R} \boldsymbol{\Phi} + \sigma_R^2 tr\left(\mathbf{R}^H \mathbf{R}\right) = E_R \tag{6}$$

where $tr\left(.\right)$ denotes the trace operator.

2) Optimization: The Lagrangian may be constructed as

$$\begin{aligned} L &= E_S \boldsymbol{\theta}^H \mathbf{R} \boldsymbol{\Phi} \boldsymbol{\Phi}^H \mathbf{R}^H \boldsymbol{\theta} - E_S \boldsymbol{\theta}^H \mathbf{R} \boldsymbol{\Phi} \\ &\quad - E_S \boldsymbol{\Phi}^H \mathbf{R}^H \boldsymbol{\theta} + \sigma_R^2 \boldsymbol{\theta}^H \mathbf{R} \mathbf{R}^H \boldsymbol{\theta} \\ &\quad + \mu \left(E_S \boldsymbol{\Phi}^H \mathbf{R}^H \mathbf{R} \boldsymbol{\Phi} + \sigma_R^2 tr\left(\mathbf{R}^H \mathbf{R}\right) - E_R\right) \end{aligned} \tag{7}$$

where μ is the Lagrange multiplier [8].

The first necessary condition for the minimum is

$$\frac{\partial L}{\partial \mathbf{R}} = \mathbf{0}$$

which leads to:

$$\mathbf{R} = \frac{E_S}{\sigma_R^2} \frac{\boldsymbol{\theta} \boldsymbol{\Phi}^H}{\left(\mu^* + \boldsymbol{\theta}^H \boldsymbol{\theta}\right)\left(1 + \frac{E_S}{\sigma_R^2} \boldsymbol{\Phi}^H \boldsymbol{\Phi}\right)} \tag{8}$$

The second necessary condition for the minimum is

$$\frac{\partial L}{\partial \mu} = 0$$

which leads to:

$$E_S \boldsymbol{\Phi}^H \mathbf{R}^H \mathbf{R} \boldsymbol{\Phi} + \sigma_R^2 tr\left(\mathbf{R}^H \mathbf{R}\right) = E_R \tag{9}$$

Solving (8) and (9) for the relay matrix, one obtains

$$\mathbf{R} = \sqrt{\frac{E_R}{\sigma_R^2}} \frac{\boldsymbol{\theta} \boldsymbol{\Phi}^H}{\sqrt{\left(\boldsymbol{\theta}^H \boldsymbol{\theta}\right)\left(\boldsymbol{\Phi}^H \boldsymbol{\Phi}\right)\left(1 + \frac{E_S}{\sigma_R^2} \boldsymbol{\Phi}^H \boldsymbol{\Phi}\right)}} \tag{10}$$

C. Application to Uniform Linear Array (ULA)

Note that for the uplink (from source to relay), we have the array steering vector as

$$\boldsymbol{\Phi} = \sqrt{\gamma_{S,R}} \acute{\boldsymbol{\Phi}} \tag{11}$$

where

$$\acute{\boldsymbol{\Phi}} \triangleq \acute{h}_{S,R} \begin{bmatrix} 1 \\ \exp\left(\frac{-j2\pi d \sin(\phi)}{\lambda}\right) \\ \vdots \\ \exp\left(\frac{-j2\pi (N-1) d \sin(\phi)}{\lambda}\right) \end{bmatrix}$$

$\gamma_{S,R}$ is the attenuation due to path-loss between the source and the relay.
$\acute{h}_{S,R}$ is the normalized channel coefficient between the source and the relay: $E\left[\left|\acute{h}_{S,R}\right|^2\right] = 1$.
N is the number of elements (antennas) of the array relay.
d is the spacing between adjacent elements in the array.
ϕ is the angle of arrival with respect to the normal to the line along which the ULA lies.
λ is the wavelength.

The array steering vector for the downlink (from relay to destination) can be written as

$$\boldsymbol{\theta} = \sqrt{\gamma_{R,D}} \acute{\boldsymbol{\theta}} \tag{12}$$

where

$$\acute{\boldsymbol{\theta}} \triangleq \acute{h}_{R,D}^* \begin{bmatrix} 1 \\ \exp\left(\frac{-j2\pi d \sin(\theta)}{\lambda}\right) \\ \vdots \\ \exp\left(\frac{-j2\pi (N-1) d \sin(\theta)}{\lambda}\right) \end{bmatrix}$$

$\gamma_{R,D}$ is the attenuation due to path-loss between the relay and the destination.
$\acute{h}_{R,D}$ is the normalized channel coefficient between the relay and the destination: $E\left[\left|\acute{h}_{R,D}\right|^2\right] = 1$.
θ is the angle of departure with respect to the normal to the line along which the ULA lies.

Fig. 1 shows the geometry of the ULA relay path.

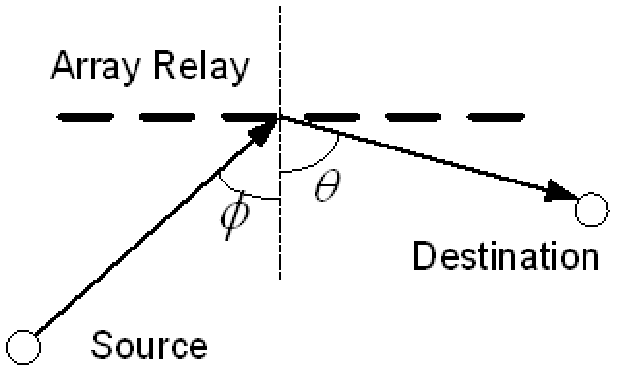

Fig. 1. The geometry of ULA relay communication path.

D. Instantaneous SNR and Maximum Spectral Efficiency of ULA Relay

It is assumed that there is no direct path between the source and the destination so the only link from the source to the destination is through the relay. The maximum instantaneous spectral efficiency of the system is given by

$$SE = \frac{1}{2}\log_2\left(1 + SNR\right) \tag{13}$$

(bps/Hz) where the instantaneous SNR is obtained from (4):

$$SNR = E_S \frac{\left|\boldsymbol{\theta}^H \mathbf{R}\boldsymbol{\Phi}\right|^2}{\sigma_R^2 \boldsymbol{\theta}^H \mathbf{R}\mathbf{R}^H \boldsymbol{\theta} + \sigma_D^2} \tag{14}$$

The factor $1/2$ is added to the spectral efficiency expression since we are transmitting one symbol over two symbols interval.

Substituting (10) in (14), (14) becomes

$$SNR = \frac{\left(\frac{E_S}{\sigma_R^2}\boldsymbol{\Phi}^H\boldsymbol{\Phi}\right)\left(\frac{E_R}{\sigma_D^2}\boldsymbol{\theta}^H\boldsymbol{\theta}\right)}{\left(\frac{E_S}{\sigma_R^2}\boldsymbol{\Phi}^H\boldsymbol{\Phi}\right)+\left(\frac{E_R}{\sigma_D^2}\boldsymbol{\theta}^H\boldsymbol{\theta}\right)+1} \tag{15}$$

which is valid for arbitrarly positioned relay antenna cluster.

Combining (11) and (12) and (15), (15) becomes

$$SNR = N\frac{\left(SNR_{S,R}\left|\acute{h}_{S,R}\right|^2\right)\left(SNR_{R,D}\left|\acute{h}_{R,D}\right|^2\right)}{\left(SNR_{S,R}\left|\acute{h}_{S,R}\right|^2\right)+\left(SNR_{R,D}\left|\acute{h}_{R,D}\right|^2\right)+\frac{1}{N}} \tag{16}$$

where

$SNR_{S,R} \triangleq \frac{E\left[||\boldsymbol{\Phi}x||^2\right]}{E\left[||\mathbf{n}_R||^2\right]} = \frac{E_S\gamma_{S,R}}{\sigma_R^2}$ is the average SNR between the source and the relay for any value of N.

$SNR_{R,D} \triangleq \frac{1}{N}\frac{E\left[\left|\boldsymbol{\theta}^H\mathbf{y}_T\right|^2\right]}{E\left[|n_D|^2\right]} = \frac{E_R\gamma_{R,D}}{\sigma_D^2}$ is the average SNR between the relay and the destination for $N = 1$.

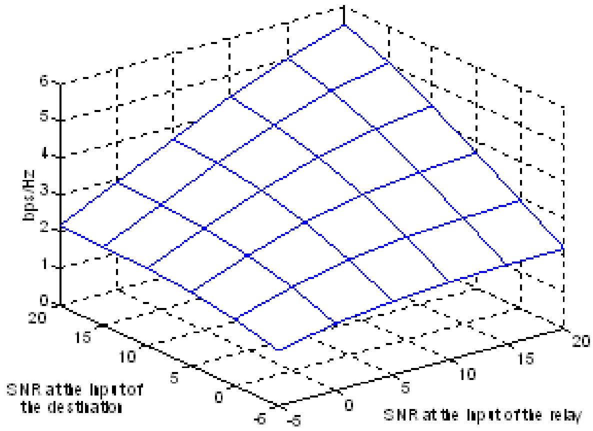

Fig. 2. Maximum spectral efficiency attainable versus $SNR_{S,R}$ and $SNR_{R,D}$ if 100 elements are employed in the array.

The factor N in front of the SNR expression (16) comes from the array gain. The additive term $\frac{1}{N}$ in the denominator of (16) comes from the fact that as N is increased, since the noise level in the reception band of the destination is constant, the relative power of noise compared to the sum signal in the band decreases. Note also that the SNR expression given by (16) does not depend on θ and ϕ. Furthermore, (16) is independent of d as long as the model described by (11) and (12) remains valid.

The array gain is defined as

$$AG \triangleq \frac{SNR}{SNR\,|_{N=1}}$$

Then it can be found that:

$$AG = N\frac{\left(SNR_{S,R}\left|\acute{h}_{S,R}\right|^2\right)+\left(SNR_{R,D}\left|\acute{h}_{R,D}\right|^2\right)+1}{\left(SNR_{S,R}\left|\acute{h}_{S,R}\right|^2\right)+\left(SNR_{R,D}\left|\acute{h}_{R,D}\right|^2\right)+\frac{1}{N}} \geq N$$

III. SIMULATION RESULTS

In order to find the ergodic maximum spectral efficiency as a function of N, $SNR_{S,R}$ and $SNR_{R,D}$, SE is averaged over 1000 realizations of $\acute{h}_{S,R}$ and $\acute{h}_{R,D}$. The normalized channel coefficients are assumed to be zero-mean Gaussian distributed complex random variables. Fig. 2 shows the maximum spectral efficiency for $N = 100$ as a function of $SNR_{S,R}$ and $SNR_{R,D}$.

Fig. 3 shows the effect of the number of elements, N, in the array on the maximum attainable spectral efficiency. The curve with circles shows the performance when $SNR_{S,R} = SNR_{R,D} = 10dB$ and the curve with crosses shows the performance when $SNR_{S,R} = SNR_{R,D} = 20dB$.

IV. COMMENTS AND CONCLUSION

Based on the results from the previous section, the following observations are made:

- The array gain is guaranteed to be at least N and increases with increasing values of N.

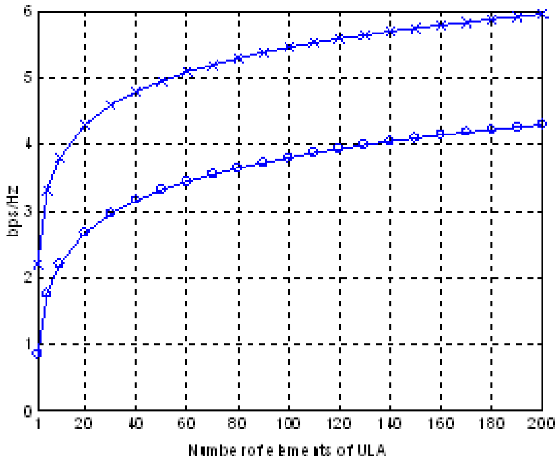

Fig. 3. Maximum attainable spectral efficiency versus N, the number of elements in the ULA. The curve with circles shows the performance when $SNR_{S,R} = SNR_{R,D} = 10dB$ and the curve with crosses shows the performance when $SNR_{S,R} = SNR_{R,D} = 20dB$.

- As the number of antenna elements of the relay is increased, the maximum spectral efficiency attainable increases.
- As can be seen from Fig. 2, the ergodic spectral efficiency attainable is maximum when $SNR_{S,R} = SNR_{R,D}$. Note that SNR definitions made in (16) include the effect of path-losses due to the distance between the source and the relay and due to the distance between the relay and the destination; the effect of the average energies available at the source and the relay outputs and the effect of the noise variances at the relay and at the destination.

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